ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 223 (2010) 619–671

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

On a random graph evolving by degrees

Boris Pittel [1]

Department of Mathematics, Ohio State University, 231 West 18th Avenue, Columbus, OH, USA

Received 14 November 2007; accepted 24 August 2009

Available online 9 September 2009

Communicated by Gil Kalai

To Irina, my wife of fifty years

Abstract

We consider a random (multi)graph growth process $\{G_m\}$ on a vertex set $[n]$, which is a special case of a more general process proposed by Laci Lovász in 2002. G_0 is empty, and G_{m+1} is obtained from G_m by inserting a new edge e at random. Specifically, the conditional probability that e joins two currently disjoint vertices, i and j, is proportional to $(d_i+\alpha)(d_j+\alpha)$, where d_i, d_j are the degrees of i, j in G_m, and $\alpha>0$ is a fixed parameter. The limiting case $\alpha=\infty$ is the Erdős–Rényi graph process. We show that whp G_m contains a unique giant component iff $c:=2m/n>c_\alpha=\alpha/(1+\alpha)$, and the size of this giant is asymptotic to $n[1-(\frac{\alpha+c^*}{\alpha+c})^\alpha]$, where $c^*<c_\alpha$ is the root of $\frac{c}{(\alpha+c)^{2+\alpha}}=\frac{c^*}{(\alpha+c^*)^{2+\alpha}}$. A phase transition window is proved to be contained, essentially, in $[c_\alpha-An^{-1/3}, c_\alpha+Bn^{-1/4}]$, and we conjecture that $1/4$ may be replaced with $1/3$. For the multigraph version, $\{MG_m\}$, we show that MG_m is connected whp iff $m\gg m_n:=n^{1+\alpha^{-1}}$. We conjecture that, for $\alpha>1$, m_n is the threshold for connectedness of G_m itself.

MSC: 05C80; 60K35

Keywords: Random graph; Multigraph; Evolution by degrees; Phase transition; Giant component

1. Introduction

In the classic Erdős–Rényi random graph process $\{G(n,m)\}_{m\geqslant 0}$, edges are added randomly, one edge at a time, to an initially empty graph $G(n,0)$ on the vertex set $[n]$. More precisely,

E-mail address: bgp@math.ohio-state.edu.

[1] Pittel's research supported in part by NSF Grants DMS-0406024 and DMS-0805996.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2009.08.015

given the graph $G(n,m)$ with m edges, location of a new $(m+1)$-th edge is chosen uniformly at random among all remaining $\binom{n}{2}-m$ vacancies. Since the pioneering work of Erdős and Rényi [17] back in 1960, there have been done numerous studies of various properties of this and other related graph-processes. The monographs by Bollobás [9] and Janson et al. [19] make it abundantly clear that by now there is a well-developed theory of random graphs, which continues to grow.

In a remarkable development, during the last several years the random graph models have been found indispensable in the statistical-combinatorial studies of evolving communication networks; see Barabási and Albert [3], Barabási et al. [4], Watts and Strogatz [42], Watts [41], Bollobás et al. [11], Bollobás and Riordan [12], Cooper and Frieze [15], Durrett [16], for instance. However, the nature of the real-life growing networks demanded substantial changes in structural-probabilistic postulates at the core of the models. It certainly was more realistic to assume that the vertex set grows as well (see though Aldous and Pittel [1] for an Erdős–Rényi graph process with "immigrating vertices"), and that the new vertex is more likely to join the old vertices with higher degrees, i.e. with larger number of neighbors in the current graph. "Preferential attachment" is a term sometimes used to describe this feature of the graph process. For a single parent case, the resulting graph (non-uniform random recursive tree) had been studied some years earlier, see Bergeron et al. [7], Mahmoud et al. [25], Pittel [32], for instance.

A two-sided version of the process above, with a fixed vertex set, has been known as a polymerization model for many years; see Flory [18], Spouge [37], Stockmayer [39,40], Whittle [43], Pittel et al. [36], Pittel and Woyczynski [34,35]. The edges (bonds) in these random graphs were inserted *or* deleted with "association–disassociation" rates dependent on the current degrees of *both* of their endvertices. The focus was on the *stationary* distribution of edges locations, and the abrupt changes of that distribution in a vicinity of the critical value of the ratio of those rates [34,35].

In Spring of 2002, Laci Lovász suggested that it might be worthwhile to study a *transitional*, time-dependent, evolution of an initially empty graph on a fixed vertex set $[n]$, in which the probability of a new edge to join two vertices, u and v, is proportional to a product $f(\deg(u))f(\deg(v))$, with a given (non-decreasing) function f, $f(0)>0$. Our goal in this paper is to study a phase transition phenomenon in this graph process for the case of a linear function $f(d)=d+\alpha$, $\alpha>0$. (Observe that, intuitively at least, the Erdős–Rényi process is the extreme case $\alpha=\infty$.) Only later did it occur to the author that the α-process was a non-stationary counterpart of the equilibrium model in [34,35], for a special choice of association–disassociation rates.

Bollobás et al. [10] had studied a time-dependent in- and out-degree distribution for a directed graph process. Both the vertex set and the edge set grew in time. As one of three possible transitions from a current state, an edge was formed between two existing vertices with probability proportional to the product of their (total) degrees. The resulting degree distribution was shown to be, in the limit, of a power-law type. (In the α-process, the degree distribution turns out to be asymptotically negative-binomial.)

Our main result is the proof of existence of three distinct stages in the evolution of the random α-graph. Which stage the graph is in is completely determined by the current average vertex degree c, i.e. the double edge density. Specifically, define $c_\alpha=\alpha/(\alpha+1)$, and let L_n stand for the largest component size. If

$$n^{1/3}(c-c_\alpha)\to-\infty,$$

however slowly, then, with high probability (whp), the graph is almost a forest of smallish trees, with L_n of order $\frac{\ln(n(c_\alpha-c)^3)}{(c-c_\alpha)^2}$ exactly, or

$$L_n = \Theta_p\left(\frac{\ln[n(c_\alpha - c)^3]}{(c - c_\alpha)^2}\right)$$

in short. If

$$n^{1/3}|c - c_\alpha| = O(1),$$

then $L_n = \Theta_p(n^{2/3})$. If

$$n^{1/4}(c - c_\alpha) \to \infty,$$

then whp the largest component is unique, of size $L_n = \Theta_p(n(c - c_\alpha))$, dwarfing every other component by a factor of order $n^{1/3}(c - c_\alpha)$ at least. By tradition, we call these three stages sub, near and supercritical.

Sending, formally, α to ∞, we recover some, not all, of the known results for the Erdős–Rényi (E–R) graph process with $c_\infty = 1$: those in [17], for c bounded away from 1; in Bollobás' breakthrough paper [8] (see also [9]) who for the first time identified the nearcritical stage; and in Łuczak [22], Łuczak et al. [24]. The main contribution of [22,24] was elimination of the logarithmic factors in Bollobás' estimates of the transition window width, and of the largest component size within the window. Though the α-process is quite different from the E–R one, the combinatorial-probabilistic ideas from [8,9,17], and [22], Łuczak [23] turn out to be remarkably robust and useful. We gratefully acknowledge our debt upfront.

We believe that, analogously to the results for the E–R graph process, the supercritical stage sets in even earlier, once $n^{1/3}(c - c_\alpha) \to \infty$, however slowly. Proving this, i.e. replacing $n^{1/4}$ with $n^{1/3}$ in the definition of that stage, may well require a deeper enumerational-analytic insight.

If and when the stated conjecture is confirmed, the α-process will join a still small club of the graph processes with the phase transition window having width of a proven order $n^{-1/3}$. It includes, in addition to the E–R graph process, the percolation processes on a random regular graph of low degree Pittel [33], Nachmias and Peres [31], and on a class of deterministic regular graphs with vertex-transitivity property, Borgs et al. [13,14].

A direct analysis of our process is hardly possible. Fortunately, as long as number of edges is linear in n, we can relax the definition of graph dynamics allowing multiple edges and loops. Of course, such a relaxation has to be rigorously justified, which is done in Section 3. The resulting multigraph process turns out to be surprisingly hospitable to enumerative techniques, just like the classic graph process. However, the advent of connectedness is much trickier, since the number of edges sufficient for the graph to be connected whp may well be superlinear in n. What we prove is that the threshold value for the number of edges in the multigraph process is $n^{1+\alpha^{-1}}$. Intuitively, this seems to indicate that, for $\alpha < 1$, whp the random graph remains disconnected almost till the moment when K_n is formed. We also conjecture that, for $\alpha > 1$, the threshold for connectedness of the random graph is the same as for the random multigraph, i.e. $n^{1+\alpha^{-1}}$.

In the next section we give a not-too-technical discussion of the (multi)graph process under study, ending it with the derivation of the limiting vertex degree and an informal explanation of why c_α is the critical vertex degree. We also provide a brief description of the remaining proofs sections.

2. Random graph process

Let $\alpha > 0$ be fixed. We start with an empty graph on the vertex set $V = [n] = \{1, \dots, n\}$. Recursively, given a graph G of vertex degree $\mathbf{d} = (d_1, \dots, d_n)$, a new edge joins two still disjoint vertices $i \neq j$ with probability proportional to $(d_i + \alpha)(d_j + \alpha)$. After μ steps we obtain a random graph $G_\alpha(n, \mu)$. The case $\alpha = \infty$ corresponds to the Erdős–Rényi random graph process $\{G(n, \mu)\}$. The sequence $\{G_\alpha(n, \mu)\}$ is a Markov process.

A multigraph counterpart is the sequence $\{MG_\alpha(n, \mu)\}$ of multigraphs, with loops and multiple edges allowed. Given a current multigraph of degree $\mathbf{d}$, the new edge joins, not necessarily disjoint, vertices $i \neq j$ with probability proportional $2(d_i + \alpha)(d_j + \alpha)$, and forms a loop $i \to i$ with probability proportional to $(d_i + \alpha)(d_i + \alpha + 1)$. The normalizing factor equals

$$\begin{aligned}
&\sum_{i<j} 2(d_i + \alpha)(d_j + \alpha) + \sum_i (d_i + \alpha)(d_i + \alpha + 1) \\
&\quad = \left(\sum_i (d_i + \alpha)\right)^2 + \sum_i (d_i + \alpha) = (2\mu + n\alpha)(2\mu + n\alpha + 1),
\end{aligned}$$

where $2\mu = \sum_i d_i$ is the current total vertex degree. So we have the Markov chain whose state space is the set of all multigraphs on the vertex set $[n]$. If the steps leading to multiple edges or loops are disallowed, then we have the Markov chain $\{G_\alpha(n, \mu)\}$.

Somewhat counterintuitively, a random graph process $\{G^*_\alpha(n, \mu)\}_{\mu \leqslant m}$, which is $\{MG_\alpha(n, \mu)\}_{\mu \leqslant m}$ conditioned on the event "$MG_\alpha(n, m)$ is simple", *does not* coincide, in distribution, with $\{G_\alpha(n, \mu)\}_{\mu \leqslant m}$. However, we will see that the appealing process $\{MG_\alpha(n, \mu)\}$ plays a key role in the asymptotic analysis of the intrinsically complex graph process $\{G_\alpha(n, \mu)\}$.

Let us have a closer look at $\{MG_\alpha(n, \mu)\}$. Consider a generic multigraph G of degree $\mathbf{d} = (d_1, \dots, d_n)$, with m edges, that has m_i loops at vertex i and $m_{ij} = m_{ji}$ parallel edges between i and j, that is,

$$d_i = 2m_i + \sum_{j \neq i} m_{ij}, \quad m = \sum_i m_i + \sum_{i<j} m_{ij}.$$

We will use $\mathbf{m} = \mathbf{m}(G)$ to denote $\{\{m_i\}, \{m_{ij}\}_{i<j}\}$. Total number of paths leading to G in the process $\{MG_\alpha(n, \mu)\}_{\mu \geqslant 0}$ after m steps is

$$\frac{m!}{\prod_i m_i! \cdot \prod_{i<j} m_{ij}!}.$$

By the definition of the chain $\{MG_\alpha(n, \mu)\}_{\mu \leqslant m}$, each of these paths has the same probability, namely

$$2^{\sum_{i<j} m_{ij}} \cdot \frac{\prod_{i=1}^n (\alpha)_{d_i}}{(n\alpha)_{2m}},$$

which is the product of m one-step transitional probabilities. (Here $(a)_b$ denotes the *rising* factorial $a(a+1)\cdots(a+b-1)$.) So

$$\mathbf{Pr}\big(MG_\alpha(n,m)=G\big)=\frac{\prod_{i=1}^n(\alpha)_{d_i}}{(n\alpha)_{2m}}\rho\big(\mathbf{m}(G)\big),$$
$$\rho(\mathbf{m}):=\frac{2^{\sum_{i<j}m_{ij}}m!}{\prod_{i=1}^n m_i!\prod_{1\leqslant i<j\leqslant n}m_{ij}!} \tag{2.1}$$

(cf. Janson et al. [20], Eqs. (1.1)–(1.3)). If G is simple then $m_i\equiv 0$, and $m_{ij}\in\{0,1\}$, so that

$$\rho(\mathbf{m})=2^m m!,$$

and (2.1) becomes

$$\mathbf{Pr}\big(MG_\alpha(n,m)=G\big)=\frac{2^m m!}{(n\alpha)_{2m}}\cdot\prod_{i=1}^n(\alpha)_{d_i}. \tag{2.2}$$

It is instructive to have a closer look at $G_\alpha^*(n,m)$, which is $MG_\alpha(n,m)$ conditioned on being simple. Denoting by $g(\mathbf{d})$ the total number of simple graphs of degree $\mathbf{d}$, we obtain

$$\mathbf{Pr}\big(MG_\alpha(n,m)\text{ is simple, }\mathbf{d}\big(MG_\alpha(n,m)\big)=\mathbf{d}\big)=\frac{2^m m!}{(n\alpha)_{2m}}\cdot\prod_{i=1}^n(\alpha)_{d_i}\cdot g(\mathbf{d}). \tag{2.3}$$

It is well known, see Bollobás [9] for instance, that

$$g(\mathbf{d})=\frac{(2m-1)!!}{\prod_{i=1}^n d_i!}\cdot F(\mathbf{d}), \tag{2.4}$$

where the fudge factor $F(\mathbf{d})\in(0,1)$. A lot of work has been done to obtain asymptotic formulas for $F(\mathbf{d})$, from Bender and Canfield [6] (for $\max_i d_i=O(1)$) to McKay and Wormald [27] (for $\max_i d_i=o(m^{1/3})$). In particular, McKay [26] showed that, for $\max_i d_i=o(m^{1/4})$,

$$F(\mathbf{d})=\exp\Big(-\eta(\mathbf{d})/2-\eta^2(\mathbf{d})/4+O\Big(m^{-1}\Big(\max_i d_i\Big)^4\Big)\Big),$$
$$\eta(\mathbf{d}):=\frac{1}{2m}\sum_{i=1}^n d_i(d_i-1). \tag{2.5}$$

Combining (2.3) and (2.4), we have

$$\mathbf{Pr}\big(MG_\alpha(n,m)\text{ is simple, }\mathbf{d}\big(MG_\alpha(n,m)\big)=\mathbf{d}\big)=\frac{(2m)!}{(n\alpha)_{2m}}\cdot\prod_{i=1}^n\frac{(\alpha)_{d_i}}{d_i!}\cdot F(\mathbf{d}). \tag{2.6}$$

Unexpectedly, leaving out the factor $F(\mathbf{d})$, we obtain a probability distribution on the set of all $\mathbf{d}$ such that $\sum_i d_i=2m$. Indeed, using the (negative) binomial theorem

$$(1-z)^{-a}=\sum_{j\geqslant 0}\frac{(a)_j}{j!}z^j,\quad |z|<1, \tag{2.7}$$

we see that

$$\sum_{\mathbf{d}} \prod_{i=1}^{n} \frac{(\alpha)_{d_i}}{d_i!} = \bigl[z^{2m}\bigr]\biggl(\sum_{d\geqslant 0} \frac{(\alpha)_d}{d!} z^d\biggr)^n$$
$$= \bigl[z^{2m}\bigr](1-z)^{-n\alpha} = \frac{(n\alpha)_{2m}}{(2m)!}. \tag{2.8}$$

So, let $\mathbf{D} = (D_1, \dots, D_n)$ denote the random vector such that

$$\mathbf{Pr}(\mathbf{D} = \mathbf{d}) = \frac{(2m)!}{(n\alpha)_{2m}} \cdot \prod_{i=1}^{n} \frac{(\alpha)_{d_i}}{d_i!}, \qquad \sum_{i=1}^{n} d_i = 2m.$$

The variables $D_1, \dots, D_n$ are "almost" independent. Here is what we mean. Let $\mathbf{Z} = (Z_1, \dots, Z_n)$ be a sequence of i.i.d. random variables such that

$$\mathbf{Pr}(Z = j) = (1-z)^{\alpha} \frac{(\alpha)_j}{j!} z^j, \qquad j \geqslant 0,$$

for some $z \in (0, 1)$. That is, Z is negative-binomially distributed with parameter α. Then, see (2.8),

$$\mathbf{Pr}\Biggl(\sum_{i=1}^{n} Z_i = 2m\Biggr) = (1-z)^{n\alpha} z^{2m} \sum_{\mathbf{d}} \prod_{i=1}^{n} \frac{(\alpha)_{d_i}}{d_i!}$$
$$= (1-z)^{n\alpha} z^{2m} \cdot \frac{(n\alpha)_{2m}}{(2m)!},$$

so that

$$\mathbf{Pr}\Biggl(\mathbf{Z} = \mathbf{d} \Bigm| \sum_{i=1}^{n} Z_i = 2m\Biggr) = \frac{\mathbf{Pr}(\mathbf{Z} = \mathbf{d})}{\mathbf{Pr}(\sum_{i=1}^{n} Z_i = 2m)}$$
$$= \frac{(1-z)^{n\alpha} z^{2m} \prod_{i=1}^{n} \frac{(\alpha)_{d_i}}{d_i!}}{\mathbf{Pr}(\sum_{i=1}^{n} Z_i = 2m)}$$
$$= \frac{(2m)!}{(n\alpha)_{2m}} \cdot \prod_{i=1}^{n} \frac{(\alpha)_{d_i}}{d_i!}$$
$$= \mathbf{Pr}(\mathbf{D} = \mathbf{d}).$$

Therefore, for every $z \in (0, 1)$, $\mathbf{D}$ has the same distribution as $\mathbf{Z}$, conditioned on $\sum_{i=1}^{n} Z_i = 2m$.

Furthermore (2.6) yields

$$\mathbf{Pr}\bigl(MG_{\alpha}(n, m) \text{ is simple}\bigr) = \mathbf{E}\bigl(F(\mathbf{D})\bigr). \tag{2.9}$$

(2.9) and (2.5) can be used to get an asymptotic formula for $\mathbf{Pr}(MG_\alpha(n,m)$ is simple) in the case $m = \Theta(n)$. Here is how. Let us select z such that

$$\mathbf{E}[Z] = c := \frac{2m}{n}.$$

Since $\mathbf{E}[Z] = \alpha z/(1-z)$, we have $z = c/(c+\alpha)$. For this choice of z, by a local limit theorem,

$$\mathbf{Pr}\left(\sum_{i=1}^{n} Z_j = 2m\right) = \Theta\bigl(n^{-1/2}\bigr).$$

Consequently, uniformly for all $A \subseteq \mathbb{N}^n$,

$$\mathbf{Pr}(\mathbf{D} \in A) = O\bigl(n^{1/2}\,\mathbf{Pr}(\mathbf{Z} \in A)\bigr).$$

Since by Chebyshev's inequality, for every $\varepsilon > 0$,

$$\begin{aligned}
&\mathbf{Pr}\left(\left|\sum_{i=1}^{n} Z_i(Z_i-1) - n\mathbf{E}\bigl[Z(Z-1)\bigr]\right| \geqslant \varepsilon n\mathbf{E}\bigl[Z(Z-1)\bigr]\right)\\
&\quad \leqslant \frac{n\,\mathrm{Var}[Z(Z-1)]}{\varepsilon^2 n^2 \mathbf{E}^2[Z(Z-1)]} \leqslant \frac{n\mathbf{E}[Z^4]}{\varepsilon^2 n^2 \mathbf{E}^2[Z(Z-1)]} = O\bigl(n^{-1}\bigr),
\end{aligned}$$

and $e^{F(\mathbf{D})} \leqslant 1$, it follows then easily that

$$\mathbf{E}\bigl[F(\mathbf{D})\bigr] \sim \exp\bigl(-\mathbf{E}\bigl[\eta(\mathbf{Z})\bigr]/2 - \mathbf{E}^2\bigl[\eta(\mathbf{Z})\bigr]/4\bigr).$$

Hence

$$\mathbf{Pr}\bigl(MG_\alpha(n,m) \text{ is simple}\bigr) \sim \exp\bigl(-(c/c_\alpha)/2 - (c/c_\alpha)^2/4\bigr), \quad c_\alpha := \frac{\alpha}{\alpha+1}. \tag{2.10}$$

Recalling that $MG_\alpha(n,m)$, conditioned on being simple, is $G^*_\alpha(n,m)$, we obtain from (2.6), (2.9), and (2.2) that

$$\begin{aligned}
&\mathbf{Pr}\bigl(\mathbf{d}\bigl(G^*_\alpha(n,m)\bigr) = \mathbf{d}\bigr) = \mathbf{Pr}(\mathbf{D} = \mathbf{d}) \cdot \frac{F(\mathbf{d})}{\mathbf{E}(F(\mathbf{D}))},\\
&\mathbf{Pr}\bigl(G^*_\alpha(n,m) = G\bigr) = \frac{2^m m!}{(n\alpha)_{2m}} \cdot \prod_{i=1}^{n} (\alpha)_{d_i} \cdot \mathbf{E}^{-1}\bigl(F(\mathbf{D})\bigr), \quad \mathbf{d} = \mathbf{d}(G),
\end{aligned} \tag{2.11}$$

for every simple graph G with m edges. Since the second probability depends on G only through its degree sequence $\mathbf{d}(G)$, we obtain that, conditioned on $\mathbf{d}(G^*_\alpha(m,n)) = \mathbf{d}$, $G^*_\alpha(m,n)$ is distributed *uniformly*. By (2.1), this is not true for $MG_\alpha(n,m)$. And it is not true for $G_\alpha(n,m)$ either.

Here is a combinatorial explanation for why the explicit product in (2.6) is a probability distribution. By (2.1),

$$\mathbf{Pr}\big(\mathbf{d}\big(MG_\alpha(n,m)\big)=\mathbf{d}\big)=\frac{\prod_{i=1}^n(\alpha)_{d_i}}{(n\alpha)_{2m}}\cdot\sum_{\substack{\mathbf{m}:\ \forall i,\\ 2m_i+\sum_{j\neq i}m_{ij}=d_i}}\rho(\mathbf{m}).$$

The sum is the total number of sequences $\mathbf{x}=(x_1,\dots,x_{2m})\in[n]^{2m}$, such that $|\{k\colon x_k=i\}|=d_i$, which is $\binom{2m}{d_1,\dots,d_n}$. For future references,

$$S_n(\mathbf{d}):=\sum_{\substack{\mathbf{m}:\ \forall i,\\ 2m_i+\sum_{j\neq i}m_{ij}=d_i}}\rho(\mathbf{m})=\binom{2m}{d_1,\dots,d_n},\quad 2m:=\sum_i d_i. \tag{2.12}$$

Indeed the generic summand counts the number of $\mathbf{x}$'s such that

$$\begin{aligned}&\big|\big\{k\colon \{x_{2k-1},x_{2k}\}=\{i,j\}\big\}\big|=m_{ij},\quad 1\leqslant i\neq j\leqslant n,\\ &\big|\{k\colon x_{2k-1}=x_{2k}=i\}\big|=m_i,\quad 1\leqslant i\leqslant n,\end{aligned}$$

so that

$$2m_i+\sum_{j\neq i}m_{ij}=\big|\{k\colon x_k=i\}\big|=d_i.$$

Therefore we obtain

$$\mathbf{Pr}\big(\mathbf{d}\big(MG_\alpha(n,m)\big)=\mathbf{d}\big)=\frac{(2m)!}{(n\alpha)_{2m}}\cdot\prod_{i=1}^n\frac{(\alpha)_{d_i}}{d_i!},\quad \sum_{i=1}^n d_i=2m. \tag{2.13}$$

Thus, distribution-wise, $\mathbf{D}$ is simply the random degree sequence of the multigraph $MG_\alpha(n,m)$.

Using (2.12) and (2.8) we arrive at

$$\mathbf{Pr}\big(d_i\big(MG_\alpha(n,m)\big)=d\big)=\binom{2m}{d}\cdot\frac{(\alpha)_d\,((n-1)\alpha)_{2m-d}}{(n\alpha)_{2m}}. \tag{2.14}$$

Letting $\alpha\to\infty$, we obtain for the Erdős–Rényi random (multi)graph $MG(n,m)$:

$$\mathbf{Pr}\big(d_i\big(MG(n,m)\big)=d\big)=\binom{2m}{d}\frac{(n-1)^{2m-d}}{n^{2m}}.$$

To continue practicing, given $k<n$ and $m_1\leqslant m$, let $P(k,m_1)$ denote the probability that there are no edges between vertices in $[k]$ and vertices in $[n]\setminus[k]$, and that the total vertex degree for the subgraph on $\{1,\dots,k\}$ is $2m_1$. Denote the $\mathbf{m}$-parameters for the generic subgraphs on $[k]$ and

$[n] \setminus [k]$ by $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively; $m_2 = m - m_1$, in particular. Using $\mathbf{m}$ for the $\mathbf{m}$-parameter of the resulting graph on $[n]$, we notice that

$$\rho(\mathbf{m}) = \frac{m!}{m_1! m_2!} \rho(\mathbf{m}_1)\rho(\mathbf{m}_2).$$

So, by (2.1), (2.12), and (2.8), we obtain

$$\begin{aligned} P(k, m_1) &= \frac{\binom{m}{m_1}}{(n\alpha)_{2m}} \sum_{|\mathbf{d}|=2m_1} S_k(\mathbf{d}) \prod_{i\in[k]} (\alpha)_{d_i} \cdot \sum_{|\mathbf{d}|=2m_2} S_{n-k}(\mathbf{d}) \prod_{i\in[n]\setminus[k]} (\alpha)_{d_i} \\ &= \frac{\binom{m}{m_1}}{(n\alpha)_{2m}} \cdot (2m_1)! \sum_{|\mathbf{d}|=2m_1} \prod_{i\in[k]} \frac{(\alpha)_{d_i}}{d_i!} \cdot (2m_2)! \sum_{|\mathbf{d}|=2m_2} \prod_{i\in[n]\setminus[k]} \frac{(\alpha)_{d_i}}{d_i!} \\ &= \binom{m}{m_1} \cdot \frac{(k\alpha)_{2m_1}((n-k)\alpha)_{2m_2}}{(n\alpha)_{2m}}. \end{aligned} \tag{2.15}$$

In particular, $P(k, 0)$, the probability that the vertices $1, \ldots, k$ are all of degree 0 is given by

$$P(k, 0) = \frac{((n-k)\alpha)_{2m}}{(n\alpha)_{2m}}. \tag{2.16}$$

Furthermore, $P(1, \mu)$, the probability that the vertex 1 is isolated and has μ loops, is given by

$$P(1, \mu) = \binom{m}{\mu} \frac{(\alpha)_{2\mu}((n-1)\alpha)_{2(m-\mu)}}{(n\alpha)_{2m}}. \tag{2.17}$$

Finally, let us evaluate the factorial moments of $d_i(MG_\alpha(n, m))$. The form of the distribution of $d_i(MG_\alpha(n, m))$ in (2.14) makes it clear that we need a closed form expression for the bivariate generating function

$$F(x, y) = \sum_{j\geqslant 0} y^j \sum_{k=0}^{j} x^k \frac{(\alpha)_k(\beta)_{j-k}}{k!(j-k)!}.$$

Here it is,

$$F(x, y) = \sum_{k\geqslant 0} (yx)^k \frac{(\alpha)_k}{k!} \cdot \sum_{\ell\geqslant 0} y^\ell \frac{(\beta)_\ell}{\ell!} = (1 - yx)^{-\alpha}(1 - y)^{-\beta}.$$

Differentiating $F(x, y)$ t times with respect to x, setting $x = 1$, and picking the coefficient by y^j, we get then

$$\sum_{k=0}^{j} \langle k\rangle_t \frac{(\alpha)_k(\beta)_{j-k}}{k!(j-k)!} = \frac{(\alpha)_t(\alpha+\beta+t)_{j-t}}{(j-t)!}, \quad t \leqslant j, \tag{2.18}$$

where $\langle k\rangle_t$ denotes the falling factorial $k(k-1)\cdots(k-t+1)$. Applying (2.18) to (2.14), we obtain

$$\mathbf{E}\big[\big\langle d_i\big(MG_\alpha(n,m)\big)\big\rangle_t\big]=\frac{\langle 2m\rangle_t(\alpha)_t}{(n\alpha)_t},\quad t\geqslant 0. \tag{2.19}$$

In particular, if $c=2m/n$ is bounded as $m,n\to\infty$, we see that

$$\mathbf{E}\big[\big\langle d_i\big(MG_\alpha(n,m)\big)\big\rangle_t\big]\sim (c/\alpha)^t(\alpha)_t,\quad t\geqslant 0.$$

So consequently, by the method of moments,

$$\mathbf{Pr}\big\{d_i\big(MG_\alpha(n,m)\big)=j\big\}\sim\frac{(\alpha)_j}{j!}\left(\frac{\alpha}{c+\alpha}\right)^{\alpha}\left(\frac{c}{c+\alpha}\right)^{j},\quad j\geqslant 0. \tag{2.20}$$

(If not for our desire to obtain the formula (2.19), we could have obtained (2.20) directly from (2.14).) Thus the limiting distribution of $d_i(MG_\alpha(n,m))$ is that of Z for a special $z=c/(c+\alpha)$.

We will prove that $MG_\alpha(n,m)$ (and $G_\alpha(n,m)$) develops a giant component when m, the number of edges exceeds $m_\alpha=c_\alpha n/2$, with c_α defined in (2.10). Here is a heuristic explanation suggested by Bollobás' pairing model [9], and by a lucid discussion of its ramifications in Durrett [16, Ch. 1]. We should expect that, for the *bounded* average vertex degree c, in a vicinity of a generic vertex $v\in[n]$ both $MG_\alpha(n,m)$ and $G_\alpha(n,m)$ look like a tree rooted at v, and that the number of direct descendants of every descendant of v has the distribution proportional to

$$\big\{(j+1)\,\mathbf{Pr}\big\{d_v\big(MG_\alpha(n,m)\big)=(j+1)\big\}\big\}_{j\geqslant 0}.$$

If so, the expected number of those descendants is

$$\frac{\sum_{j\geqslant 0}j(j+1)\,\mathbf{Pr}\{d_v(MG_\alpha(n,m))=(j+1)\}}{\sum_{j\geqslant 0}(j+1)\,\mathbf{Pr}\{d_v(MG_\alpha(n,m))=(j+1)\}}=\frac{\mathbf{E}[\langle d_v(MG_\alpha(n,m))\rangle_2]}{\mathbf{E}[d_v(MG_\alpha(n,m))]}.$$

By (2.19), this ratio is

$$\frac{\langle 2m\rangle_2 n\alpha(\alpha)_2}{\langle n\alpha\rangle_2 2m\alpha}=\big(1+O\big(n^{-1}\big)\big)\frac{c}{c_\alpha},$$

and thus, for $c<c_\alpha$, the tree for a generic vertex v is likely to be small, and there is no giant component. Alternatively, we could have arrived, again informally, at the same conclusion by using an essentially equivalent condition discovered by Molloy and Reed [28,29] as being necessary for existence of a giant component in a random graph with a given degree sequence.

We hope the reader feels by now that potentially $MG_\alpha(n,m)$ is as amenable to the enumerative-asymptotic techniques as the classic $G(n,m)$.

In addition to being a tractable random graph model, $MG_\alpha(n,m)$ can be used to deal with the intrinsically harder $G_\alpha(n,m)$. Here is why. For $m=O(n)$, we will show in the next section that every likely (unlikely) event for $MG_\alpha(n,m)$ is a likely (unlikely) event for $G_\alpha(n,m)$ itself. So we can afford working with $MG_\alpha(n,m)$ only, at least as long as m, the number of edges, is linear in n, the number of vertices. And this is where in the graph process $\{G_\alpha(n,\mu)\}$ the transition from a forest of small trees to a graph with a giant component takes place.

The rest of the paper is organized as follows. In Section 3 we prove the approximation property, which allows us to work with $MG_\alpha(n,m)$ instead of $G_\alpha(n,m)$ for the sparse case $m = O(n)$ in Sections 4–6. In Section 4, we compute the moments of the counts of tree-components and unicyclic components, discuss the related enumerational-analytic issues, and provide a more substantial insight into the phase transition phenomenon. In Section 5 we bound the expected counts of components with cycles. In Section 6 we state and prove the main result, Theorem 1, on the largest component size in the three ranges of the average vertex degree c. The proof is split accordingly into three parts, given in Sections 6.1, 6.2 and 6.3, respectively. In Section 7 we prove Theorem 2, which states that the random multigraph $MG_\alpha(n,m)$ is connected with high probability iff $m \gg n^{1+\alpha^{-1}}$.

3. Approximating $G_\alpha(n,m)$ by $MG_\alpha(n,m)$

For a graph G, we denote its edge set by $\mathcal{E}(G)$, and the degree of a vertex r by $\deg_r(G)$. By the definition, given $G_\alpha(n,t)$, the next $(t+1)$-th edge joins two vertices i, j,

$$(i,j) \notin \mathcal{E}_t := \mathcal{E}\big(G_\alpha(n,t)\big) \bigcup_{i\in[n]} \big\{(i,i)\big\},$$

with the (conditional) probability

$$\frac{2D_i(t)D_j(t)}{\sum_{u,v:\ (u,v)\notin\mathcal{E}_t} D_u(t)D_v(t)}, \qquad D_r(t) := \deg_r\big(G_\alpha(n,t)\big) + \alpha. \tag{3.1}$$

Notice that

$$\begin{aligned}\sum_{u,v:\ (u,v)\notin\mathcal{E}_t} D_u(t)D_v(t) &= \Big(\sum_{u\in[n]} D_u(t)\Big)^2 - \sum_{u,v:\ (u,v)\in\mathcal{E}_t} D_u(t)D_v(t)\\ &= (2t+n\alpha)^2 - \sum_{u,v:\ (u,v)\in\mathcal{E}_t} D_u(t)D_v(t).\end{aligned} \tag{3.2}$$

Our task is to show that, with high probability, the sum in (3.2) is of order $O(n)$ for all $t \leqslant m$, if $m = O(n)$. (If so, the normalizing factor in (3.1) is

$$(2t+n\alpha)^2\big(1+O\big(n^{-1}\big)\big),$$

and thus the product of these factors for $t \leqslant m = O(n)$ is essentially independent on the random sequence $\{G_\alpha(n,t)\}_{t\leqslant m}$.) The first step is the following

Lemma 1. *Let $m = O(n)$.*

(a) *For every $K > 0$, there exists $L = L(K) > 0$ such that*

$$\mathbf{Pr}\Big\{\max_u D_u(m) \leqslant L\ln n\Big\} \geqslant 1 - n^{-K}.$$

(b) *For every* $\delta \in (0, 1/2)$,

$$\mathbf{Pr}\Big\{\max_u D_u(m) \leqslant n^{\delta}\Big\} \geqslant 1 - \exp\big(-\sigma n^{\delta}\big), \quad \sigma = \sigma(\delta) > 0;$$

thus $\max_u D_u(m)$ *does not exceed a fractional power of n with probability subexponentially close to* 1.

Note. Here and elsewhere we use σ, with or without adornments, to denote positive constants whose actual values are immaterial for us. Occasionally we will use a notation $B \leqslant_b C$ to indicate that $B = O(C)$ uniformly over a specified range of parameters that determine the values of B and C.

Proof of Lemma 1. We will denote the expectation and the probability conditioned on $G_\alpha(n,t)$ by $\mathbf{E}[\cdot \mid \circ]$ and $\mathbf{Pr}\{\cdot \mid \circ\}$, respectively. By (3.1) and (3.2),

$$\begin{aligned} \mathbf{E}\big[D_i(t+1) \,\big|\, \circ\big] &= D_i(t) + \frac{2D_i(t)\sum_{j:\ (i,j)\notin\mathcal{E}_t} D_j(t)}{(2t+n\alpha)^2 - \sum_{u,v:\ (u,v)\in\mathcal{E}_t} D_u(t)D_v(t)} \\ &\leqslant D_i(t) + \frac{2D_i(t)(2t+n\alpha)}{(2t+n\alpha)^2 - \sum_{u,v:\ (u,v)\in\mathcal{E}_t} D_u(t)D_v(t)}, \end{aligned} \tag{3.3}$$

as

$$\sum_{j:\ (i,j)\notin\mathcal{E}_t} D_j(t) \leqslant \sum_{j\in[n]} D_j(t) = 2t + n\alpha.$$

Let $\delta = \delta(n) > 0$ be such that $\limsup \delta < \frac{1}{2}$. Introduce a stopping time T,

$$T = \min\Big\{t \leqslant m\colon\ \max_i D_i(t) > n^{\delta}\Big\},$$

if such t exist, and set $T = m+1$ otherwise. Let $t < T$. Since δ is bounded away from $1/2$, the sum in the denominator in (3.3) is $O(n^{1+2\delta}) = o(n^2)$. Therefore

$$\begin{aligned} \mathbf{E}\big[D_i(t+1) \,\big|\, \circ\big] &\leqslant D_i(t) + \frac{2D_i(t)}{2t+n\alpha}\big(1 + O\big(n^{-1+2\delta}\big)\big) \\ &= \frac{2(t+1)+n\alpha}{2t+n\alpha} D_i(t) + O\big(n^{-2+3\delta}\big), \end{aligned}$$

or

$$\mathbf{E}\big[X_i(t+1) \,\big|\, \circ\big] \leqslant X_i(t) + O\big(n^{-3+3\delta}\big), \quad X_i(t) := \frac{D_i(t)}{2t+n\alpha}. \tag{3.4}$$

From the definition of $X_i(t)$ it follows also that, for all $t < T$,

$$X_i(t+1) - X_i(t) = O\big(n^{-1}\big) + O\big(n^{-2}D_i(t)\big) = O\big(n^{-1}\big) + O\big(n^{-2+\delta}\big); \tag{3.5}$$

here $O(n^{-1})$ comes from the case

$$\deg_i(t+1) = \deg_i(t) + 1,$$

and we know that

$$\mathbf{Pr}\{\deg_i(t+1) = \deg_i(t) + 1 \mid \circ\} = O\big(n^{-1+\delta}\big). \tag{3.6}$$

What remains of the argument is an adapted version of the well-known proof of Azuma's inequality (Alon and Spencer [2], Wormald [44]), for martingales and sub(super)martingales.

Pick a small $\varepsilon > 0$. By (3.4), for $t < T$,

$$\begin{aligned}
\mathbf{E}\big[e^{\varepsilon n X_i(t+1)} \mid \circ\big] &= e^{\varepsilon n X_i(t)}\mathbf{E}\big[e^{\varepsilon n(X_i(t+1)-X_i(t))} \mid \circ\big] \\
&\leqslant e^{\varepsilon n X_i(t)}\big(1 + \varepsilon n\mathbf{E}\big[X_i(t+1) - X_i(t) \mid \circ\big] + O\big(\varepsilon^2 n^2 \mathbf{E}\big[\big(X_i(t+1) - X_i(t)\big)^2 \mid \circ\big]\big)\big) \\
&\leqslant e^{\varepsilon n X_i(t)}\big(1 + O\big(\varepsilon n^{-2+3\delta}\big) + O\big(\varepsilon^2 n^{-1+\delta}\big) + O\big(\varepsilon^2 n^{-2+2\delta}\big)\big) \\
&= e^{\varepsilon n X_i(t)}\big(1 + O\big(\varepsilon^2 n^{-1+\delta}\big)\big) \\
&= e^{\varepsilon n X_i(t)}\exp\big(O\big(\varepsilon^2 n^{-1+\delta}\big)\big).
\end{aligned}$$

(To bound the second moment we have used (3.5),

$$(a+b)^2 \leqslant 2\big(a^2 + b^2\big),$$

and (3.6).) Thus, for $t < T$,

$$\mathbf{E}\big[e^{\varepsilon n X_i(t+1)} \mid \circ\big] \leqslant e^{\varepsilon n X_i(t)}\exp\big(O\big(\varepsilon^2 n^{-1+\delta}\big)\big). \tag{3.7}$$

For $t \geqslant T$, redefine $X_i(t) \equiv X_i(T)$. Clearly (3.7) holds then for all $t \leqslant m$. Therefore, as $X_i(0) = n^{-1}$, and $m = O(n)$,

$$\mathbf{E}\big[e^{\varepsilon n X_i(t)}\big] \leqslant e^{\varepsilon}\exp\big(O\big(\varepsilon^2 m n^{-1+\delta}\big)\big) \leqslant e^{\varepsilon}\exp\big(\gamma\varepsilon^2 n^{\delta}\big)$$

for some constant γ. Then, by Markov inequality,

$$\mathbf{Pr}\big\{X_i(t) \geqslant 2\gamma\varepsilon n^{\delta-1}\big\} \leqslant e^{\varepsilon}\exp\big(-\gamma\varepsilon^2 n^{\delta}\big).$$

Consequently

$$\begin{aligned}
\mathbf{Pr}\Big\{\max_{t\leqslant T}\max_i X_i(t) \geqslant 2\gamma\varepsilon n^{\delta-1}\Big\} &\leqslant \sum_i\sum_{t\leqslant m}\mathbf{Pr}\big\{X_i(t) \geqslant 2\gamma\varepsilon n^{\delta-1}\big\} \\
&= O\big[\exp\big(-\gamma\varepsilon^2 n^{\delta} + 2\ln n\big)\big] \\
&= O\left[\exp\left(-\frac{\gamma\varepsilon^2}{2}n^{\delta}\right)\right] \to 0,
\end{aligned}$$

with the last equality holding under the condition

$$n^\delta \geqslant \frac{4}{\gamma\varepsilon^2}\ln n. \tag{3.8}$$

Since $T < m+1$ means

$$\max_i X_i(T) = \frac{\max_i D_i(T)}{2T+n\alpha} > \frac{n^\delta}{2m+n\alpha},$$

we see then that

$$\begin{aligned}
\mathbf{Pr}\Big\{\max_i D_i(m) \leqslant n^\delta\Big\} &= \mathbf{Pr}\{T = m+1\}\\
&\geqslant \mathbf{Pr}\Big\{\max_i X_i(T) \leqslant \frac{n^\delta}{2m+n\alpha}\Big\}\\
&\geqslant \mathbf{Pr}\Big\{\max_i X_i(T) < 2\gamma\varepsilon n^{\delta-1}\Big\}\\
&\geqslant 1 - O\bigg[\exp\bigg(-\frac{\gamma\varepsilon^2}{2}n^\delta\bigg)\bigg],
\end{aligned} \tag{3.9}$$

if

$$2\gamma\varepsilon n^{\delta-1} \leqslant \frac{n^\delta}{2m+n\alpha} \quad\Longleftrightarrow\quad \frac{2\gamma\varepsilon}{n} \leqslant \frac{1}{2m+n\alpha}.$$

Since $m = O(n)$, the last condition is met if $\varepsilon > 0$ is chosen sufficiently small. As for the condition (3.8), it is definitely met if δ is bounded away from 0. Thus, for every such δ, $\max_i D_i(m)$ is n^δ at most, with probability subexponentially close to 1.

Finally set

$$\delta = \delta(n) = \frac{\ln\ln n}{\ln n} + \frac{\ln A}{\ln n}, \quad A > 0.$$

In this case $\delta \to 0$, but

$$n^\delta = A\ln n \geqslant \frac{4}{\gamma\varepsilon^2}\ln n,$$

for $A \geqslant 4/(\gamma\varepsilon^2)$. And then, by (3.9),

$$\begin{aligned}
\mathbf{Pr}\Big\{\max_i D_i(m) \leqslant A\ln n\Big\} &\geqslant 1 - O\bigg[\exp\bigg(-A\frac{\gamma\varepsilon^2}{2}\ln n\bigg)\bigg]\\
&= 1 - O\big(n^{-A\sigma}\big), \quad \sigma := \frac{\gamma\varepsilon^2}{2}.
\end{aligned} \tag{3.10}$$

The relations (3.9), (3.10) are equivalent to the statement. $\square$

By Lemma 1(b), for the sum in (3.2) we have: given $K > 0$, there exists $L(K) > 0$ such that

$$\mathbf{Pr}\left\{\sum_{u,v:\ (u,v)\in\mathcal{E}_t} D_u(t)D_v(t) \leqslant Ln\ln^2 n\right\} \geqslant 1 - n^{-K}. \tag{3.11}$$

Using the weaker n^{δ}-bound (Lemma 1(a)) we will prove that the sum in (3.11) is of order n, with a probability subexponentially close to 1.

Lemma 2. *Let $m = O(n)$. There exist positive constants σ^* and $\hat{\sigma}$ such that*

$$\mathbf{Pr}\left\{\sum_{u,v:\ (u,v)\in\mathcal{E}_m} D_u(m)D_v(m) \leqslant \sigma^* n\right\} \geqslant 1 - O\bigl(e^{-\hat{\sigma}n^{1/5}}\bigr). \tag{3.12}$$

Proof. First of all,

$$\begin{aligned}
\sum_{u,v:\ (u,v)\in\mathcal{E}_t} D_u(t)D_v(t) &\leqslant \frac{1}{2}\sum_{u,v:\ (u,v)\in\mathcal{E}_t}\bigl(D_u^2(t)+D_v^2(t)\bigr) = \sum_{u,v:\ (u,v)\in\mathcal{E}_t} D_u^2(t)\\
&\leqslant \sum_{u\in[n]} D_u^2(t)\bigl(D_u(t)+1\bigr) \leqslant 2Q(t);\\
Q(t) &:= \sum_{u\in[n]} D_u^3(t).
\end{aligned}$$

So it suffices to show that, with high probability, $Q(m) = O(m)$. Like in the proof of Lemma 1, let T be the first $t \leqslant m$ such that $\max_i D_i(t) > n^{\delta}$, where $\delta \in (0, 1/2)$ is fixed and set $T = m+1$ if no such t exists. We showed that

$$\mathbf{Pr}\{T < m+1\} = O\bigl(e^{-\sigma n^{\delta}}\bigr), \quad \sigma = \sigma(\delta) > 0. \tag{3.13}$$

Let $t < T$. Then

$$\begin{aligned}
Q(t) &\leqslant (2t+n\alpha)\max_i D_i^2(t) \leqslant \sigma_1 n^{1+2\delta},\\
Q(t+1) - Q(t) &\leqslant 2\max_i\bigl[\bigl(D_i(t)+1\bigr)^3 - D_i^3(t)\bigr]\\
&\leqslant 6\bigl(1+\alpha^{-1}\bigr)^2 \max_i D_i^2(t) \leqslant \sigma_2 n^{2\delta}.
\end{aligned} \tag{3.14}$$

Further, as in (3.3),

$$\begin{aligned}
&\mathbf{E}\bigl[D_i^3(t+1)\bigm|\circ\bigr]\\
&\quad\leqslant D_i^3(t) + \bigl(\bigl(D_i(t)+1\bigr)^3 - D_i^3(t)\bigr)\cdot\frac{2D_i(t)(2t+n\alpha)}{(2t+n\alpha)^2 - \sum_{u,v:\ (u,v)\in\mathcal{E}_t} D_u(t)D_v(t)}\\
&\quad\leqslant D_i^3(t) + 6\bigl(1+\alpha^{-1}\bigr)^2 D_i^2(t)\cdot\frac{D_i(t)(2t+n\alpha)}{(2t+n\alpha)^2 - 2\sigma_1 n^{1+2\delta}}.
\end{aligned}$$

Summing for $i \in [n]$, we obtain then

$$\begin{aligned}
\mathbf{E}\big[Q(t+1)\,\big|\,\circ\big] &\leqslant Q(t) + 6\big(1+\alpha^{-1}\big)^2 \frac{2t+n\alpha}{(2t+n\alpha)^2 - 2\sigma_1 n^{1+2\delta}} \cdot Q(t) \\
&\leqslant Q(t)\left(1 + \frac{7(1+\alpha^{-1})^2}{2t+n\alpha}\right), \quad \text{(as } 1+2\delta < 2\text{)}, \\
&\leqslant Q(t)(1+\sigma_3/n). \qquad (3.15)
\end{aligned}$$

Introduce

$$\hat{Q}(t) = (1+\sigma_3/n)^{-t} Q(t), \quad t < T.$$

Clearly

$$\hat{Q}(t) = \Theta\big(Q(t)\big), \quad t \leqslant m; \qquad \hat{Q}(0) = Q(0) = n\alpha^3.$$

By (3.14) and (3.15), for $t < T$,

$$\mathbf{E}\big[\hat{Q}(t+1)\,\big|\,\circ\big] \leqslant \hat{Q}(t), \quad \big|\hat{Q}(t+1) - \hat{Q}(t)\big| \leqslant \sigma_4 n^{2\delta}. \qquad (3.16)$$

Setting $Q(t) \equiv Q(T)$, $\hat{Q}(t) \equiv \hat{Q}(T)$ for $t \geqslant T$, we extend (3.16) to all $t \leqslant m$. Applying the Azuma-type inequality for supermartingales with bounded increments (see [44]) to $\hat{Q}(t)$, we obtain

$$\begin{aligned}
\mathbf{Pr}\big\{Q(m) - Q(0)(1+\sigma_3/n)^m \geqslant n\big\} &= \mathbf{Pr}\left\{\hat{Q}(m) - \hat{Q}(0) \geqslant \frac{n}{(1+\sigma_3/n)^m}\right\} \\
&\leqslant \exp\left(-\frac{n^2(1+\sigma_3/n)^{-2m}}{2m(\sigma_4 n^{2\delta})^2}\right) \\
&= \exp\big(-\sigma_5 n^{1-4\delta}\big), \qquad (3.17)
\end{aligned}$$

a meaningful bound for $\delta < 1/4$. Since $Q(t)$ does not decrease with t, we obtain that

$$Q(T) < n\alpha^3 e^{\sigma_3 m/n} + n \leqslant \sigma_6 n,$$

with probability approaching 1 subexponentially fast. Using (3.10) and (3.17), we conclude:

$$\begin{aligned}
\mathbf{Pr}\Big\{\big(Q(m) \leqslant \sigma_6 n\big) \cap \Big(\max_i D_i(m) < n^\delta\Big)\Big\} &= \mathbf{Pr}\big\{\big(Q(m) < \sigma_6 n\big) \cap (T = m+1)\big\} \\
&\geqslant 1 - O\big(e^{-\sigma n^\delta} + e^{-\sigma_5 n^{1-4\delta}}\big).
\end{aligned}$$

Of course, the sigmas do depend on δ, but they are well defined for every fixed $\delta \in (0, 1/2)$. The best δ is $1/5$ since it equalizes the powers of n in the two remainder terms. Thus

$$\mathbf{Pr}\Big\{\big(Q(m) \leqslant \sigma_6 n\big) \cap \Big(\max_i D_i(m) < n^{1/5}\Big)\Big\} \geqslant 1 - O\big(e^{-\hat{\sigma} n^{1/5}}\big).$$

And, of course, on the event $\{\max_i D_i(m) < n^{1/5}\}$ we have

$$Q(m) = \sum_{i\in[n]} D_i^3(m). \qquad \square$$

Corollary 3. *Let $m = O(n)$. Uniformly for all sets $\mathcal{G}$ of graphs on $[n]$ with m edges,*

$$\mathbf{Pr}\big\{G_\alpha(n,m)\in\mathcal{G}\big\} \leqslant_b \exp\big(-\hat\sigma n^{1/5}\big) + \mathbf{Pr}\big\{MG_\alpha(n,m)\in\mathcal{G}\big\}.$$

Proof. We begin with

$$\begin{aligned}&\big\{G_\alpha(n,m)\in\mathcal{G}\big\} \subseteq \big\{G_\alpha(n,m)\in B\big\} + \big\{G_\alpha(n,m)\in B^c\big\}\cap\big\{G_\alpha(n,m)\in\mathcal{G}\big\},\\ &B := \left\{G \text{ with } \big|\mathcal{E}(G)\big| = m\colon \sum_{u,v\colon (u,v)\notin\mathcal{E}(G)} D_u(G)D_v(G) > \sigma^* n\right\}.\end{aligned}$$

Here σ^* comes from (3.12), so that

$$\mathbf{Pr}\big\{G_\alpha(n,m)\in B\big\} \leqslant \exp\big(-\hat\sigma n^{1/5}\big).$$

Let $G\in B^c$ and $G\in\mathcal{G}$. Let $\mathbf{G}$ denote a generic sequence of $m+1$ nested graphs $G_0, G_1, \dots, G_m$, with $G_0 = \emptyset$, $G_m = G$, such that each G_μ is obtained by inserting an edge into $G_{\mu-1}$. Notice upfront that the total number of those sequences $\mathbf{G}$ is $m!$. By the definition of $\{G_\alpha(n,\mu)\}$,

$$\begin{aligned}\mathbf{Pr}\big\{G_\alpha(n,m)=G\big\} &= \sum_{\mathbf{G}} \mathbf{Pr}\left\{\bigcap_{\mu=0}^m \big(G(n,\mu)=G_\mu\big)\right\}\\ &= \sum_{\mathbf{G}}\prod_{\mu=1}^m \mathbf{Pr}\big\{G(n,\mu)=G_\mu \,\big|\, G(n,\mu-1)=G_{\mu-1}\big\}.\end{aligned}$$

Let (i_μ, j_μ) be the new edge in G_μ. Here, by (3.1), (3.2), and the condition of $G\in B^c$,

$$\mathbf{Pr}\big\{G(n,\mu)=G_\mu \,\big|\, G(n,\mu-1)=G_{\mu-1}\big\} = \frac{2(\deg_{i_\mu}(G_{\mu-1})+\alpha)(\deg_{j_\mu}(G_{\mu-1})+\alpha)}{(2(\mu-1)+n\alpha)^2(1+O(n^{-1}))},$$

uniformly for all $\mathbf{G}$ and $\mu\leqslant m$. Therefore, as $m = O(n)$,

$$\begin{aligned}&\mathbf{Pr}\big\{G_\alpha(n,m)=G\big\}\\ &\quad\leqslant_b \frac{2^m}{\prod_{\mu=1}^m(2(\mu-1)+n\alpha)^2}\sum_{\mathbf{G}}\prod_{\mu=1}^m\big[\big(\deg_{i_\mu}(G_{\mu-1})+\alpha\big)\big(\deg_{j_\mu}(G_{\mu-1})+\alpha\big)\big]\\ &\quad= \frac{2^m}{\prod_{\mu=1}^m(2(\mu-1)+n\alpha)^2}\left(\prod_{i\in[n]}(\alpha)_{\deg_i(G)}\right)\sum_{\mathbf{G}} 1\\ &\quad= \frac{2^m m!}{\prod_{\mu=1}^m(2(\mu-1)+n\alpha)^2}\prod_{i\in[n]}(\alpha)_{\deg_i(G)}.\end{aligned}$$

Further,

$$\big(2(\mu-1)+n\alpha\big)^2=\big(2(\mu-1)+n\alpha\big)\big(2(\mu-1)+1+n\alpha\big)\big(1+O\big(n^{-1}\big)\big),$$

and so

$$\prod_{\mu=1}^{m}\big(2(\mu-1)+n\alpha\big)^2\geqslant\sigma_1\prod_{\mu=1}^{n}(2\mu-2+n\alpha)(2\mu-1+n\alpha)=\sigma_1(n\alpha)_{2m}.$$

Hence, uniformly for $G\in B^c\cap\mathcal{G}$,

$$\begin{aligned}\mathbf{Pr}\big\{G_\alpha(n,m)=G\big\}&\leqslant_b\frac{2^m m!}{(n\alpha)_{2m}}\prod_{i\in[n]}(\alpha)_{\deg_i(G)}\\&=\mathbf{Pr}\big\{MG_\alpha(n,m)=G\big\},\end{aligned}\tag{3.18}$$

see (2.2). □

Notes. 1. The definition of $G^*_\alpha(n,m)$ as $MG_\alpha(n,m)$ conditioned on being simple, and (3.18) taken together imply that we also have

$$\mathbf{Pr}\big\{G_\alpha(n,m)\in\mathcal{G}\big\}\leqslant_b\exp\big(-\hat{c}n^{1/5}\big)+\mathbf{Pr}\big\{G^*_\alpha(n,m)\in\mathcal{G}\big\}.\tag{3.19}$$

2. The power of Corollary 3 is in that it allows to prove rarity of the events for $G_\alpha(n,m)$ by showing rarity of the corresponding events for $MG_\alpha(n,m)$. As we will start seeing in the next section, the computations for $MG_\alpha(n,m)$ are surprisingly manageable.

4. Tree components and the unicyclic components

We assume that $m=\Theta(n)$, so that $c=2m/n$, the average vertex degree of $G_\alpha(n,m)$, and of $MG_\alpha(n,m)$, is bounded away from zero and infinity. For $k\geqslant1$, let E_k denote the expected number of tree components of $MG_\alpha(n,m)$ with k vertices, and for $k^1,k^2\geqslant1$ let E_{k^1,k^2} denote the expected number of ordered pairs of distinct tree components, with k^1 and k^2 vertices respectively.

Lemma 4.

(a)
$$E_k=2^{k-1}\frac{\langle n\rangle_k\langle m\rangle_{k-1}((n-k)\alpha)_{2(m-k+1)}}{(n\alpha)_{2m}}\cdot\frac{\alpha^k(k(\alpha+1))_{k-2}}{k!},\tag{4.1}$$

where

$$\langle\mu\rangle_\nu=\mu(\mu-1)\cdots(\mu-\nu+1),\qquad(\alpha+1)_{-1}:=\alpha^{-1},$$

and

$$E_{k_1,k_2} = \frac{\langle n\rangle_k \langle m\rangle_{k-2}((n-k)\alpha)_{2(m-k+2)}}{(n\alpha)_{2m}} \cdot \prod_{t=1}^{2} 2^{k_t-1} \frac{\alpha^{k_t}(k_t(\alpha+1))_{k_t-2}}{k_t!} \quad (k := k_1+k_2). \tag{4.2}$$

(b) *Consequently, if* $m = \Theta(n)$ *and* $k = O(n^{3/4})$, *then*

$$\begin{aligned}
E_k &= n\frac{\alpha^k c^{k-1}}{(\alpha+c)^{2(k-1)}}\left(\frac{\alpha}{\alpha+c}\right)^{k\alpha} \cdot \frac{(k(\alpha+1))_{k-2}}{k!}\, e^{\Psi_{n,k}},\\
\Psi_{n,k} &= \beta\frac{k^2}{n} + \beta_1\frac{k^3}{n^2} + O\big(k/n + k^4/n^3\big),\\
\beta &= \beta(c) := -\frac{\alpha+1}{2c(\alpha+c)}(c-c_\alpha)(c-2),\\
\beta_1 &= \beta_1(c) := -\frac{1}{6(\alpha+c)^2}\big(f(c) - 6\alpha^2 - 12\alpha - 8\big),\\
f(c) &:= \frac{(c^2+4)(\alpha+c)^2}{c^2} + \alpha c(2\alpha+c),\\
c_\alpha &:= \frac{\alpha}{\alpha+1}.
\end{aligned} \tag{4.3}$$

(c) *If* $k_1, k_2 = o(n)$, *then*

$$\begin{aligned}
\frac{E_{k_1,k_2}}{E_{k_1}E_{k_2}} &= e^{\Psi_{n,\mathbf{k}}},\\
\Psi_{n,\mathbf{k}} &= \frac{2k_1k_2}{n}\big[\beta(c) + O(k/n)\big] + O(k/n), \quad k = k_1+k_2.
\end{aligned} \tag{4.4}$$

Note. The way the formula for $\beta(c)$ is written is to make obvious an important fact, namely that, as c grows, $\beta(c)$ first changes its sign, from negative to positive, at $c = c_\alpha$. As we will see, this is one of the technical reasons why c_α is the critical value of the average vertex degree. As for $\beta_1(c)$, what will matter only is that $\beta_1(c) < 0$ for $c \leqslant c_\alpha$ and a little bit beyond c_α, at least. (Indeed, for $f(\cdot)$ in the definition of $\beta_1(c)$, we have

$$f'(x) = 2(\alpha+x)\big[(1+\alpha) - 4\alpha x^{-3}\big].$$

Hence $f(x)$ is decreasing on $(0, (4c_\alpha)^{1/3}) \supset (0, c_\alpha)$, and it is easy to check that

$$f(c_\alpha) > 6\alpha^2 + 12\alpha + 8.)$$

Proof of Lemma 4. (a) By symmetry, $E_k = \binom{n}{k}P_k$, where P_k is the probability that $[k]$ is the vertex set of a tree component. Consider $k \geqslant 2$, such that $2(k-1) \leqslant 2m$. Given positive integers $d_1, \dots, d_k$ such that $\sum_{j\in[k]} d_j = 2(k-1)$, there are

$$\binom{k-2}{d_1-1, \dots, d_k-1} \tag{4.5}$$

of possible trees with the vertex degrees $d_1, \dots, d_k$, Moon [30], Stanley [38]. Let T be a generic tree on $[k]$, $\mathcal{T}_k$ be the set of all k^{k-2} such trees, and G be a generic multigraph on $[n] \setminus [k]$, with the $\mathbf{m}$-parameters $\mathbf{m}_1 = \mathbf{m}(\mathcal{T})$ and $\mathbf{m}_2 = \mathbf{m}(G)$. In particular,

$$\rho(\mathbf{m}_1) = 2^{k-1}(k-1)!.$$

Then, arguing as in (2.15) and using (4.5),

$$\begin{aligned} P_k &= \frac{\binom{m}{k-1}}{(n\alpha)_{2m}} 2^{k-1}(k-1)! \sum_{T \in \mathcal{T}_k} \prod_{i \in [k]} (\alpha)_{d_i(T)} \cdot \sum_G \rho\big(\mathbf{m}(G)\big) \prod_{i \in [n] \setminus [k]} (\alpha)_{d_i(G)} \\ &= \frac{\binom{m}{k-1}}{(n\alpha)_{2m}} 2^{k-1}(k-1)!(k-2)! \cdot \Sigma(k) \cdot \big((n-k)\alpha\big)_{2(m-k+1)}, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \Sigma(k) &:= \sum_{|\mathbf{d}|=2(k-1)} \prod_{i \in [k]} \frac{(\alpha)_{d_i}}{(d_i - 1)!} \\ &= \big[x^{2(k-1)}\big]\left(\sum_{d \geqslant 1} \frac{(\alpha)_d}{(d-1)!} x^d\right)^k \\ &= \big[x^{2(k-1)}\big]\left(x \frac{d}{dx} \sum_{d \geqslant 1} \frac{(\alpha)_d}{d!} x^d\right)^k \\ &= \big[x^{2(k-1)}\big]\left(x \frac{d}{dx} (1-x)^{-\alpha}\right)^k \\ &= \alpha^k \big[x^{2(k-1)}\big]\big(x^k (1-x)^{-k(\alpha+1)}\big) \\ &= \alpha^k \frac{(k(\alpha+1))_{k-2}}{(k-2)!}. \end{aligned} \tag{4.7}$$

Hence, for $k \geqslant 2$,

$$E_k = 2^{k-1} \frac{\langle n \rangle_k \langle m \rangle_{k-1} ((n-k)\alpha)_{2(m-k+1)}}{(n\alpha)_{2m}} \cdot \frac{\alpha^k (k(\alpha+1))_{k-2}}{k!}. \tag{4.8}$$

As for $k = 1$, using (2.16), we have

$$E_1 = n \,\mathbf{Pr}\big\{d_1\big(MG_\alpha(n,m)\big) = 0\big\} = n \frac{((n-1)\alpha)_{2m}}{(n\alpha)_{2m}},$$

which is covered by (4.8) too, if we define $(\alpha+1)_{-1} = \alpha^{-1}$.

Analogously

$$E_{k_1,k_2} = \frac{\langle n \rangle_k \langle m \rangle_{k-2} ((n-k)\alpha)_{2(m-k+2)}}{(n\alpha)_{2m}} \cdot \prod_{t=1}^{2} 2^{k_t - 1} \frac{\alpha^{k_t} (k_t(\alpha+1))_{k_t - 2}}{k_t!}. \tag{4.9}$$

This proves the part (a) of Lemma 4. The part (b) follows then via application of standard asymptotic techniques to the n, m-dependent factors in (4.8), which we omit for brevity.

Later we will need a similar, but less precise formula: if $\mathcal{M} = \Theta(N)$, $\nu, \mu = O(N^{3/4})$, $\mu \leqslant 2\nu$, then, denoting $\xi = 2\mathcal{M}/N$,

$$\frac{\langle N\rangle_\nu \langle \mathcal{M}\rangle_\mu ((N-\nu)\alpha)_{2(\mathcal{M}-\mu)}}{(N\alpha)_{2\mathcal{M}}} = N^{\nu-\mu}\left(\frac{x(\xi)}{2\alpha}\right)^\nu \times \exp\left(\beta(\xi)\frac{\nu^2}{N} + O\big(|\mu-\nu|+1\big) + O\big(\nu^3/N^2\big)\right), \tag{4.10}$$

where

$$x(\xi) := \frac{\xi\alpha^{\alpha+1}}{(\alpha+\xi)^{\alpha+2}}, \tag{4.11}$$

and $\beta(\cdot)$ is defined in (4.3).

The proof of the part (c) is given in Appendix A. $\square$

Note. It is worth noticing that (4.7) formally means that

$$\sum_{k\geqslant 1}\frac{x^k}{k!}\sum_{T\in\mathcal{T}_k}\prod_{i\in[k]}(\alpha)_{d_i(T)} = \sum_{k\geqslant 1}(\alpha x)^k\frac{(k(\alpha+1))_{k-2}}{k!}, \tag{4.12}$$

which can be thought of as the formula for an *α-analogue* of the exponential generating function of the unrooted trees. Shortly we will find a closed form expression for the series on the right.

Let us discuss, *informally*, the ramifications of Lemma 4. For $\alpha = 1$, (4.3) becomes

$$E_k \sim n(1+c)^{-1}\left(\frac{c}{(1+c)^3}\right)^{k-1}\frac{(3(k-1))!}{k!(2k-1)!}, \quad k = o\big(n^{1/2}\big).$$

So for the expected total *size* of all such tree components we have

$$\sum_{k=o(n^{1/2})} kE_k \sim n(1+c)^{-1}F\big(c(1+c)^{-3}\big),$$
$$F(u) := \sum_{j\geqslant 0} u^j\frac{1}{2j+1}\binom{3j}{2j}, \tag{4.13}$$

if c is such that $c(1+c)^{-3}$ is less than the convergence radius of F. And now comes a surprise: $f_j := \frac{1}{2j+1}\binom{3j}{2j}$ happens to be the number of all *ternary* plane rooted trees with j internal, i.e. non-leaf vertices; $f_0 = 1$, by definition. Ternary means that all internal vertices have out-degree 3. (See Stanley [38, vol. 2, Exercise 5.45].) In particular,

$$F(x) := \sum_{j=0}^{\infty} x^j f_j$$

satisfies

$$F(x) = 1 + xF^3(x), \tag{4.14}$$

and the series converges for $|x| \leqslant 4/27$. This is the point where

$$\frac{\partial}{\partial y}\left(y - 1 - xy^3\right)\Big|_{y=F(x)} = 0,$$

and consequently

$$F(4/27) = 3/2 < \infty.$$

On the other hand,

$$\frac{c}{(1+c)^3} \leqslant \frac{4}{27}, \quad \forall c \geqslant 0,$$

since

$$\sup\left\{\frac{z}{(1+z)^3}\colon z \geqslant 0\right\} = \frac{1/2}{(1+1/2)^3} = \frac{4}{27}.$$

So

$$\sum_{k=o(n^{1/2})} kE_{nk} \sim \frac{n}{1+c} F\left(\frac{c}{(1+c)^3}\right), \quad \forall c > 0,$$

and $F(c/(1+c)^3) < \infty$. Therefore at $c = 1/2$

$$\sum_{k=o(n^{1/2})} kE_{nk} \sim n\frac{F(4/27)}{1+1/2} = n.$$

That is, for c close to $1/2$, almost all vertices are in the tree components of size much less than $n^{1/2}$. We should expect this to be true for all $c < 1/2$ as well, which means that

$$F\left(\frac{c}{(1+c)^3}\right) = 1 + c, \quad \forall c \leqslant \frac{1}{2}.$$

And yes, this is so, because

$$1 + c = 1 + \frac{c}{(1+c)^3}(1+c)^3,$$

and $c(1+c)^{-3}$ increases from 0 to $4/27$ as c increases from 0 to $1/2$. This argument also shows that, for $c > 1/2$,

$$F\left(\frac{c}{(1+c)^3}\right) = 1 + c^*,$$

where c^* is uniquely determined by

$$\frac{c^*}{(1+c^*)^3} = \frac{c}{(1+c)^3}, \quad c^* \in (0, 1/2).$$

For $c > 1/2$, the function $(1+c^*)/(1+c)$ is below 1. By analogy with the Erdős–Rényi graph $G(n,m)$, we should expect then that once c exceeds $1/2$, the random multigraph $MG_1(n,m)$ whp has a giant component of size close to

$$n\left(1 - \frac{F(c/(1+c)^3)}{1+c}\right) = n\left(1 - \frac{1+c^*}{1+c}\right) = n\,\frac{c-c^*}{1+c}.$$

What happens for the general $\alpha > 0$? Introduce $F(x)$ as the solution of

$$F(x) = 1 + xF^{2+\alpha}(x), \tag{4.15}$$

compare with (4.14). The corresponding series converges for

$$|x| \leqslant x_\alpha := \frac{(1+\alpha)^{1+\alpha}}{(2+\alpha)^{2+\alpha}}, \qquad F(x_\alpha) = \frac{2+\alpha}{1+\alpha}.$$

Using the Lagrange inversion formula, one easily obtains that, for $\beta > 0$,

$$\left[x^r\right]F^\beta(x) = \frac{\beta}{r!}\big(\beta + 1 + r(1+\alpha)\big)_{r-1}, \quad r \geqslant 0. \tag{4.16}$$

Comparing this with the first line in (4.3), we see that the general version of (4.13) should be

$$\sum_{k=o(n^{1/2})} kE_k \sim n\left(\frac{\alpha}{\alpha+c}\right)^\alpha \cdot F^\alpha\big(x(c)\big), \quad x(c) := \frac{c\alpha^{\alpha+1}}{(\alpha+c)^{2+\alpha}}. \tag{4.17}$$

Furthermore, the right side expression in (4.12) becomes $\int_0^x F^\alpha(\alpha y)\,dy$. Using the substitution $\alpha y = (z-1)z^{-2-\alpha}$ (see (4.15)), after a series of elementary integrations we obtain

$$\int_0^x F^\alpha(\alpha y)\,dy = xF^\alpha(\alpha x) - \frac{\alpha^2x^2}{2}F^{2\alpha+2}(\alpha x). \tag{4.18}$$

Therefore (4.12) becomes

$$\sum_{k\geqslant 1}\frac{x^k}{k!}\sum_{T\in\mathcal{T}_k}\prod_{i\in[k]}(\alpha)_{d_i(T)} = xF^\alpha(\alpha x) - \frac{\alpha^2x^2}{2}F^{2\alpha+2}(\alpha x). \tag{4.19}$$

Let $\mathcal{T}_k^\rho$ denote the set of all *rooted* trees on $[k]$. Applying xd/dx to both sides of (4.12), we infer that

$$\sum_{k\geqslant 1}\frac{x^k}{k!}\sum_{T\in\mathcal{T}_k^\rho}\prod_{i\in[k]}(\alpha)_{d_i(T)}=x\frac{d}{dx}\int_0^x F^\alpha(\alpha y)\,dy=xF^\alpha(\alpha x). \tag{4.20}$$

The identities (4.15), (4.19)–(4.20) are the α-analogues of the classic identities

$$\begin{aligned}
&T(x)=xe^{T(x)},\\
&\sum_{k\geqslant 1}\frac{x^k}{k!}k^{k-2}=T(x)-\frac{1}{2}T^2(x),\\
&\sum_{k\geqslant 1}\frac{x^k}{k!}k^{k-1}=T(x)
\end{aligned} \tag{4.21}$$

($|x|\leqslant e^{-1}$), see [20,29].

Returning to $MG_\alpha(n,m)$, notice that the argument $x(c)$ of F in (4.17) attains its supremum at

$$c_\alpha=\frac{\alpha}{1+\alpha}\quad\Longrightarrow\quad x(c_\alpha)=\frac{(1+\alpha)^{1+\alpha}}{(2+\alpha)^{2+\alpha}}=x_\alpha.$$

So, for $c=c_\alpha$, the right side expression in (4.17) becomes

$$n\left(\frac{\alpha}{\alpha+c_\alpha}\right)^\alpha\cdot\left(\frac{2+\alpha}{1+\alpha}\right)^\alpha=n.$$

Therefore, for $c<c_\alpha$, we must have

$$\sum_{k=o(n^{1/2})}kE_k\sim n,$$

i.e.

$$F\left(\frac{c\alpha^{\alpha+1}}{(\alpha+c)^{2+\alpha}}\right)=1+\frac{c}{\alpha},\quad\forall c\leqslant c_\alpha,$$

which indeed follows from (4.15). Then, for $c>c_\alpha$,

$$\begin{aligned}
&F\left(\frac{c\alpha^{\alpha+1}}{(\alpha+c)^{2+\alpha}}\right)=1+\frac{c^*}{\alpha},\quad \text{i.e. } F\bigl(x(c)\bigr)=1+\frac{c^*}{\alpha},\\
&\frac{c^*\alpha^{\alpha+1}}{(\alpha+c^*)^{2+\alpha}}=\frac{c\alpha^{\alpha+1}}{(\alpha+c)^{2+\alpha}},\quad \text{i.e. } x\bigl(c^*\bigr)=x(c)\quad\bigl(c^*\in(0,c_\alpha)\bigr).
\end{aligned} \tag{4.22}$$

(As $c^*=c$ for $c\leqslant c_\alpha$, (4.22) holds then for all c.) Thus we are led to believe that, once c exceeds c_α, whp the random graph $MG_\alpha(n,m)$ has a giant component, with size asymptotic to

$$n\left[1-\left(\frac{\alpha}{\alpha+c}\right)^\alpha\cdot F^\alpha\left(\frac{c\alpha^{\alpha+1}}{(\alpha+c)^{2+\alpha}}\right)\right]=n\left[1-\left(\frac{\alpha+c^*}{\alpha+c}\right)^\alpha\right]. \tag{4.23}$$

Note. The second line in (4.22) effectively means that in the postcritical stage $c > c_\alpha$ the total size of the tree components is asymptotic to that in the subcritical stage for the average vertex degree $c^* = c^*(c) < c_\alpha$.

As a partial check, let $\alpha \to \infty$. Then the factor of n in (4.23) is asymptotic to

$$1 - e^{-c} \cdot F^\alpha\big(\big(1 + o(1)\big)ce^{-c}/\alpha\big),$$

where

$$F\big(\big(1 + o(1)\big)ce^{-c}/\alpha\big) = 1 + \big(1 + o(1)\big)\frac{ce^{-c}}{\alpha}F^\alpha\big(\big(1 + o(1)\big)ce^{-c}/\alpha\big).$$

Consequently

$$F^\alpha\big(\big(1 + o(1)\big)ce^{-c}/\alpha\big) \sim \exp\big[ce^{-c}F^\alpha\big(\big(1 + o(1)\big)ce^{-c}/\alpha\big)\big],$$

so that

$$F^\alpha\big(\big(1 + o(1)\big)ce^{-c}/\alpha\big) \sim \big(ce^{-c}\big)^{-1}T\big(ce^{-c}\big),$$

where $T(x)$ is the tree function defined in (4.21). So the coefficient by n in (4.23) becomes

$$1 - c^{-1}T\big(ce^{-c}\big),$$

just what it has to be for the Erdős–Rényi random graph $G(n, m)$!.

In the next sections we will give the exact formulations and the full proofs of these claims, and also analyze a nearcritical case of c close to c_α. We will do this by combining the ideas from Bollobás [8,9], Łuczak [22], Łuczak et al. [24], Łuczak [23], and extending the enumerational techniques discussed in this section.

Note. According to (3.19), all the rare (likely) events for $G^*_\alpha(n, m)$ are rare (likely) for $G_\alpha(n, m)$ too, and $G^*_\alpha(n, m)$, conditioned on its degree sequence, is uniformly distributed. So if one can show that whp the random degree sequence of $G^*_\alpha(n, m)$ meets the "well-behavior" conditions in Molloy and Reed [28,29], then the basic formula (4.30) can also be arrived at, even in a shorter way, as a special case of their general formula for the giant component size in a random graph distributed uniformly on the set of all graphs with a given degree sequence. However, a verification of the required property of $G^*_\alpha(n, m)$, namely that whp the (weighted) random counts of vertices by degree are uniformly close to their expected values, looks quite difficult though. In fact, in a literal sense, this sufficient condition may not be even true for $G^*_\alpha(n, m)$. This, and our desire also to study in detail the two other phases, subcritical ($c < c_\alpha$) and nearcritical ($|c - c_\alpha| \to 0$), dictated our approach.

As another illustration of enumerational richness of $MG_\alpha(n, m)$, let us compute the expected number of all unicyclic components.

Unlike trees, there is no product formula for the number of unicyclic components with a given degree sequence. So the counterpart of (4.6) is the following. Let U be a generic unicyclic graph on the vertex set $[k]$, and let $d_i(U)$ denote the degree of the vertex $i \in [k]$. Then U has k edges, so $\sum_{i=1} d_i(U) = 2k$. Let $E(U)$ denote the expected number of unicyclic components of $MG_\alpha(n, m)$, each being isomorphic to U under the order preserving bijection between its vertex set and $[k]$. Analogously to (4.6), and to the n-dependent factor $ne^{\Psi_{n,k}}$ in (4.3), for $3 \leqslant k = O(n^{2/3})$,

$$E(U) = \binom{n}{k} \frac{\binom{m}{k}}{(n\alpha)_{2m}} 2^k k! \big((n-k)\alpha\big)_{2(m-k)} \prod_{i \in [k]} (\alpha)_{d_i(U)}$$

$$= e^{\hat{\Psi}_{n,k}} \frac{(x(c)/\alpha)^k}{k!} \prod_{i=1}^{k} (\alpha)_{d_i(U)};$$

$$\hat{\Psi}_{n,k} := \beta \frac{k^2}{n} + O\big(k/n + k^3/n^2\big); \tag{4.24}$$

for the second equality we have used (4.10)–(4.11). (Notice that, unlike (4.3), the factor n is out.) Our task then is to evaluate

$$S_k := \sum_{U \in \mathcal{U}_k} \prod_{i=1}^{k} (\alpha)_{d_i(U)}, \tag{4.25}$$

where $\mathcal{U}_k$ is the set of all unicyclic graphs on $[k]$.

Each U consists of a cycle of length $j \geqslant 3$ and a forest of j trees rooted at the j cyclic vertices. Given j, let $(V_1, \ldots, V_j)$ be a partition of $[k]$ into j non-empty sets, and denote $v_t = |V_t|$. Given such a partition, there are $v_1 \cdots v_j$ ways to select a vertex from each V_t as a cyclic vertex. Once the cyclic vertices are chosen, there are $(j-1)!/2$ ways to form an undirected cycle. And for each t and $\delta_1, \ldots, \delta_{v_t}$, with $\delta_1 + \cdots + \delta_{v_t} = 2(v_t - 1)$, there are $\binom{v_t - 2}{\delta_1 - 1, \ldots, \delta_{v_t} - 1}$ ways to choose a tree on V_t such that the root (cyclic) vertex has degree δ_1, and the remaining (index-ordered) vertices have degrees $\delta_2, \ldots, \delta_{v_t}$. Let $S(V_1, \ldots, V_j)$ stand for the subsum of S_k for Us that induce the partition $(V_1, \ldots, V_j)$ of $[k]$. Denoting $v_t = |V_t|$, we have then

$$S(V_1, \ldots, V_j) = \frac{(j-1)!}{2} \prod_{t=1}^{j} v_t \left(\sum_{\delta_1, \ldots, \delta_{v_t}} (\alpha)_{\delta_1 + 2} \prod_{s=2}^{v_t} (\alpha)_{\delta_s} \binom{v_t - 2}{\delta_1 - 1, \ldots, \delta_{v_t} - 1} \right).$$

The t-th factor here is $(\alpha)_2 = \alpha(\alpha+1)$ if $v_t = 1$, and if $v_t > 1$ then it is

$$v_t (v_t - 2)! \big[x^{2(v_t - 1)}\big] \left\{ \left(\sum_{\delta} \frac{(\alpha)_{\delta+2}}{(\delta-1)!} x^{\delta} \right) \left(\sum_{\delta} \frac{(\alpha)_{\delta}}{(\delta-1)!} x^{\delta} \right)^{v_t - 1} \right\}$$

$$= v_t (v_t - 2)! \alpha(\alpha+1) \big[x^{2(v_t - 1)}\big] \left\{ \left(x \frac{d}{dx} (1-x)^{-\alpha-2} \right) \left(x \frac{d}{dx} (1-x)^{-\alpha} \right)^{v_t - 1} \right\}$$

$$= v_t (v_t - 2)! \alpha^{v_t} (\alpha+1)(\alpha+2) \big[x^{v_t - 2}\big] (1-x)^{-v_t(\alpha+1) - 2}$$

$$= v_t \alpha^{v_t} (\alpha+1)(\alpha+2) \big(v_t(\alpha+1) + 2\big)_{v_t - 2}.$$

Since

$$(\alpha+3)_{-1} = \big((\alpha+2)+1\big)_{-1} = (\alpha+2)^{-1},$$

the last formula works for $v_t = 1$ as well. Using (4.25) and the last two identities, we obtain

$$\begin{aligned}
S_k &= \sum_{j,\, V_1 \cup \cdots \cup V_j = [k]} S(V_1, \ldots, V_j) \\
&= \sum_j \frac{(j-1)!}{2j!} \sum_{v_1 + \cdots + v_j = k} \binom{k}{v_1, \ldots, v_j} \prod_{t=1}^{j} \bigl[v_t \alpha^{v_t} (\alpha+1)_2 \bigl(v_t(\alpha+1)+2 \bigr)_{v_t - 2} \bigr] \\
&= \sum_{j \geqslant 3} \frac{k! \alpha^k ((\alpha+1)_2)^j}{2j} \cdot \Sigma_{jk}.
\end{aligned}$$

Here

$$\begin{aligned}
\Sigma_{jk} &= \sum_{v_1 + \cdots + v_j = k} \prod_{t=1}^{j} \frac{(v_t(\alpha+1)+2)_{v_t - 2}}{(v_t - 1)!} \\
&= \bigl[x^k\bigr] \biggl(\sum_{v \geqslant 1} \frac{(v(\alpha+1)+2)_{v-2}}{(v-1)!} x^v \biggr)^j \\
&= \bigl[x^{k-j}\bigr] \Biggl(\sum_{r=0}^{\infty} \frac{(r(\alpha+1)+\alpha+3)_{r-1}}{r!} x^r \Biggr)^j \\
&= (\alpha+2)^{-j} \bigl[x^{k-j}\bigr] \Biggl((\alpha+2) \sum_{r=0}^{\infty} \frac{(r(\alpha+1)+\alpha+3)_{r-1}}{r!} \, x^r \Biggr)^j \\
&= (\alpha+2)^{-j} \bigl[x^{k-j}\bigr] \bigl(F^{\alpha+2}(x) \bigr)^j = (\alpha+2)^{-j} \bigl[x^{k-j}\bigr] F^{j(\alpha+2)}(x);
\end{aligned}$$

for the last equality we have used (4.16) with $\beta = \alpha + 2$. Therefore

$$\begin{aligned}
S_k &= k! \alpha^k \sum_{j \geqslant 3} \frac{(\alpha+1)^j}{2j} \cdot \bigl[x^{k-j}\bigr] F^{j(\alpha+2)}(x) \\
&= k! \alpha^k \bigl[x^k\bigr] \sum_{j \geqslant 3} \frac{(\alpha+1)^j}{2j} \bigl(x F^{\alpha+2}(x) \bigr)^j \\
&= k! \alpha^k \bigl[x^k\bigr] \sum_{j \geqslant 3} \frac{(\alpha+1)^j}{2j} \bigl(F(x) - 1 \bigr)^j \\
&= k! \alpha^k \bigl[x^k\bigr] \frac{1}{2} \biggl(\ln \frac{1}{1-z} - z - \frac{z^2}{2} \biggr) \bigg|_{z = (\alpha+1)(F(x)-1)}.
\end{aligned}$$

Thus we have proved

$$\begin{aligned}
&\sum_{k\geqslant 3}\frac{x^k}{k!}\sum_{U\in\mathcal{U}_k}\prod_{i=1}^{k}(\alpha)_{d_i(U)}\\
&\quad=\sum_{k\geqslant 3}(x\alpha)^k\left[x^k\right]\frac{1}{2}\left(\ln\frac{1}{1-z}-z-\frac{z^2}{2}\right)\Big|_{z=(\alpha+1)(F(x)-1)}\\
&\quad=\frac{1}{2}\left(\ln\frac{1}{1-z}-z-\frac{z^2}{2}\right)\Big|_{z=(\alpha+1)(F(\alpha x)-1)}\\
&\quad=\frac{1}{2}\left(\ln\frac{1}{1-z}-z-\frac{z^2}{2}\right)\Big|_{z=(\alpha)_2 xF^{\alpha+2}(\alpha x)}.
\end{aligned}\tag{4.26}$$

In the limit $\alpha\to\infty$, this becomes the well-known formula for the exponential generating function of $\{u_k\}_{k\geqslant 3}$, $u_k=|\mathcal{U}_k|$,

$$\sum_{k\geqslant 3}x^k\frac{u_k}{k!}=\frac{1}{2}\left(\ln\frac{1}{1-T(x)}-T(x)-\frac{T^2(x)}{2}\right).$$

And, neglecting the factor $e^{\hat{\Psi}_{n,k}}$ in (4.24) (which can be justified for a fixed $c\neq c_\alpha$),

$$\begin{aligned}
\sum_{k\geqslant 3,\,U\in\mathcal{U}_k}E(U)&\sim\sum_k x^k(c)\big[x^k\big]\frac{1}{2}\left(\ln\frac{1}{1-z}-z-\frac{z^2}{2}\right)\Big|_{z=(\alpha+1)(F(x)-1)}\\
&=\frac{1}{2}\left(\ln\frac{1}{1-z}-z-\frac{z^2}{2}\right)\Big|_{z=(\alpha+1)(F(x(c))-1)}\\
&=\frac{1}{2}\left(\log\frac{1}{1-c^*/c_\alpha}-c^*/c_\alpha-\frac{(c^*/c_\alpha)^2}{2}\right),
\end{aligned}$$

since, by (4.22),

$$(\alpha+1)\big[F\big(x(c)\big)-1\big]=(\alpha+1)\frac{c^*}{\alpha}=\frac{c^*}{c_\alpha}.$$

Letting $\alpha\to\infty$, we recover a well-known formula for the Erdős–Rényi random graph $G(n,m)$, [17]:

$$\sum_{U\in\mathcal{U}_k}E(U)\sim\frac{1}{2}\log\frac{1}{1-c^*}-\frac{c^*}{2}-\frac{(c^*)^2}{4}\quad(c\neq c_\infty=1).$$

5. Multicyclic components

To study the supercritical phase $c>c_\alpha$ we will also need upper bounds for the expected numbers of components of the random multigraph $MG_\alpha(n,m)$, with size below $n^{2/3}$, containing more than one cycle. A key ingredient in both [7,8] and [23] was Bollobás' strong bound for $C(k,\ell)$, the number of connected graphs with k vertices and $\ell\in(k,2k]$ edges:

$$C(k,\ell) \leqslant \left(\frac{A}{t}\right)^{t/2} k^{k+(3t-1)/2}, \quad t := \ell - k, \tag{5.1}$$

$A > 0$ being an absolute constant. (5.1) extends a sharp asymptotic formula for $t = o(k^{1/3})$ due to Wright [45], in which $A = (12e)^{-1}$. Later Łuczak [23] found that, within a low degree polynomial factor, Bollobás' bound holds for any $A > (12e)^{-1}$. The idea of [23] was that, given $N > k$ and $p \in (0,1)$, $E_{k,\ell}(N,p)$ the expected number of the (k,ℓ) components in the Bernoulli random graph $G(N, \mathbf{Pr}(\text{edge} = p))$, is at most N/k, and so

$$E_{k,\ell}(N,p) = \binom{N}{k} p^{\ell} q^{\binom{k}{2} - \ell + k(N-k)} C(k,\ell) \leqslant \frac{N}{k}. \tag{5.2}$$

The bound (5.1), with any $A > (12e)^{-1}$ but with an additional factor coming from N/k, follows by selecting the best N and p, which are $N = k^{3/2}/t$, and $p \sim N^{-1}$. In that range, $E_{k,\ell}(N,p)$ is not small either, and so N/k differs from $E_{k,\ell}(N,p)$ by only a polynomial factor.

We will need an upper bound of an α-analogue of $C(k,\ell)$, which we define as

$$C_\alpha(k,\ell) = \sum_{\substack{\text{connected } G:\\ V(G)=[k],\, |E(G)|=\ell}} \rho\big(\mathbf{m}(G)\big) \prod_{i=1}^{k} (\alpha)_{d_i(G)}, \tag{5.3}$$

with $\rho(\mathbf{m}(G))$ defined in (2.1). We obtain it by adapting Łuczak's embedding idea. (For a simple graph G, $\rho(\mathbf{m}(G)) = 2^{m(G)} m(G)!$, $m(G)$ being the number of edges; so a better candidate for an α-analogue of $C(k,\ell)$ is $C_\alpha(k,\ell)/2^{\ell}\ell!$.) Introduce the random *multi*graph $MG_\alpha(N,\mathcal{M})$ ($N > k$, $\mathcal{M} > \ell$), and let $E_{k,\ell}(N,\mathcal{M})$ denote the expected number of (k,ℓ)-components in $MG_\alpha(N,\mathcal{M})$. Then, arguing as in (2.15), we obtain

$$E_{k,\ell}(N,\mathcal{M}) = \binom{N}{k}\binom{\mathcal{M}}{\ell} C_\alpha(k,\ell) \frac{((N-k)\alpha)_{2(\mathcal{M}-\ell)}}{(N\alpha)_{2\mathcal{M}}} \leqslant \frac{N}{k}, \tag{5.4}$$

which yields an upper bound on $C_\alpha(k,\ell)$, dependent on $N, \mathcal{M}$. Plugging this bound into the corresponding expression for $E_{k,\ell} := E_{k,\ell}(n,m)$, we arrive at

$$E_{k,\ell}(n,m) \leqslant \frac{N}{k} \frac{\langle n\rangle_k \langle m\rangle_\ell ((n-k)\alpha)_{2(m-\ell)}}{(n\alpha)_{2m}} \cdot \frac{(N\alpha)_{2\mathcal{M}}}{\langle N\rangle_k \langle \mathcal{M}\rangle_\ell ((N-k)\alpha)_{2(\mathcal{M}-\ell)}}. \tag{5.5}$$

In Section 6, our study of the nearcritical phase will enable us to eliminate the N/k factor in the bound (5.5). (The counterpart of this device eliminates the factor N/k from Łuczak's bound as well.) The resulting bound for $E_{k,\ell}(n,m)$ will be a key tool for analysis of the supercritical phase.

Keeping N/k in place for now, let us see what we can get from (5.5) by choosing N and $\mathcal{M}$ properly.

Lemma 5. *Let $k, \ell \leqslant n^{2/3}$, and $t := \ell - k \in (0,k]$. There exists a constant $A > 0$ such that*

$$E_{k,\ell}(n,m) \leqslant_b \frac{k^{1/2}}{t^{1/6}} \left(\frac{A}{t^{1/6}} \frac{k^{3/2}}{n}\right)^{t} \left(\frac{x(c)}{x(c_\alpha)}\right)^{k} e^{\beta(c)k^2/n}. \tag{5.6}$$

Also, if $k \leqslant n^{2/3}$ *then*

$$E_{k,k}(n,m) \leqslant_b k^{1/2}\left(\frac{x(c)}{x(c_\alpha)}\right)^k e^{\beta(c)k^2/n}. \tag{5.7}$$

Proof. Set $N = \lceil k^{3/2}/t^{1/6} \rceil$ and $\mathcal{M} = \lceil Nc_\alpha/2 \rceil$. The estimate (4.10)–(4.11) is applicable to both the second and the third fraction in (5.5), because $k, \ell \leqslant n^{3/4}$, $\ell \leqslant 2k$, and

$$\frac{k^4}{N^3}, \frac{\ell^4}{N^3} \leqslant_b \left(\frac{t}{k}\right)^{1/2} \leqslant 1.$$

Since

$$\frac{k^3}{N^2} \leqslant_b t^{1/3}, \qquad \left|\beta\left(\frac{2\mathcal{M}}{N}\right)\right|\frac{k^2}{N} \leqslant_b \frac{k^2}{N^2} \leqslant_b \frac{t^{1/3}}{k},$$

we obtain then

$$E_{k,\ell}(n,m) \leqslant_b \frac{k^{1/2}}{t^{1/6}}\left(\frac{k^{3/2}}{t^{1/6}n}\right)^t \left(\frac{x(c)}{x(c_\alpha)}\right)^k \cdot \exp\left(\beta(c)\frac{k^2}{n} + O(t)\right),$$

and (5.6) follows. To prove (5.7), we set $N = \lceil k^{3/2} \rceil$, and keep $\mathcal{M} = \lceil Nc_\alpha/2 \rceil$. $\square$

6. Largest component

Let L_n denote the size of a largest component in $G_\alpha(n,m)$. We will show that the likely order of magnitude of L_n depends on whether $c < c_\alpha$ (subcritical case), $c - c_\alpha \to 0$ (nearcritical case), or $c > c_\alpha$ (supercritical case). Here is a precise statement.

Theorem 1. *Define*

$$\omega = \omega(n) := n^{1/3}|c - c_\alpha|.$$

(a) *Subcritical case. Suppose* $c = c(n) < c_\alpha$ *is such that* $\omega \to \infty$ *however slowly. Then there exist constants* $0 < \lambda_1 < \lambda_2 < \infty$ *such that*

$$\lim_{n\to\infty} \mathbf{Pr}\left\{\lambda_1 \frac{\ln[n(c_\alpha - c)^3]}{(c - c_\alpha)^2} \leqslant L_n \leqslant \lambda_2 \frac{\ln[n(c_\alpha - c)^3]}{(c - c_\alpha)^2}\right\} = 1. \tag{6.1}$$

(b) *Nearcritical case. Suppose that* $\omega = O(1)$. *Let* $L_{n,k}$ *denote the size of a* k*-th largest component; so* $L_{n,1} = L_n$. *Then*

$$L_{n,k} = \Theta_p\left(n^{2/3}\right), \quad k \geqslant 1, \tag{6.2}$$

i.e.

$$\lim_{n\to\infty} \mathbf{Pr}\left\{\lambda^{-1} n^{2/3} \leqslant L_{n,k} \leqslant \lambda n^{2/3}\right\} = 1, \quad k \geqslant 1,$$

if $\lambda \to \infty$ *however slowly.*

(c) *Supercritical case. Suppose that $c = O(1)$ is such that $c > c_\alpha$ and*

$$\lim_{n\to\infty} n^{1/4}(c - c_\alpha) = \infty. \tag{6.3}$$

Then

$$L_n = \bigl(1 + o_p(1)\bigr)n\left[1 - \left(\frac{\alpha + c^*}{\alpha + c}\right)^{\alpha}\right], \tag{6.4}$$

where c^* is the root of

$$\frac{x\alpha^{\alpha+1}}{(\alpha + x)^{2+\alpha}} = \frac{c\alpha^{\alpha+1}}{(\alpha + c)^{2+\alpha}}, \quad x \in (0, c_\alpha), \tag{6.5}$$

and $o_p(1)$ stands for a random variable which converges to zero in probability. A second largest component is smaller, whp, by a factor of order $n^{1/3}(c - c_\alpha)$, at least.

Notes. 1. We conjecture that, as in the Erdős–Rényi graph process, the supercritical stage begins earlier, when $n^{1/3}(c - c_\alpha) \to \infty$.

2. By Corollary 3, it suffices to prove the corresponding claim for $MG_\alpha(n, m)$.

6.1. *Proof of Theorem 1(a)*

Pick a constant $A > 0$ and set

$$k_n = \left\lceil \frac{A \ln[n(c_\alpha - c)^3]}{(c_\alpha - c)^2} \right\rceil. \tag{6.1.1}$$

The condition $\omega \to \infty$ implies that

$$\lim_{n\to\infty} \frac{(c_\alpha - c)k_n^2}{n} = 0, \qquad \lim_{n\to\infty} \frac{k_n^3}{n^2} = 0.$$

Introduce S_n, the expected total size of all tree components, each of size k_n, at most, i.e.

$$S_n = \sum_{k \leqslant k_n} kE_k,$$

E_k being the expected number of tree components of size k. By (4.3) in Lemma 4(b), and (4.16),

$$\begin{aligned} kE_k &= n\left(\frac{\alpha}{\alpha + c}\right)^{\alpha} \bigl(x(c)^{k-1}\bigl[x^{k-1}\bigr]F^{\alpha}(x)\bigr)e^{\Psi_{n,k}}, \\ x(c) &:= \frac{c\alpha^{\alpha+1}}{(\alpha + c)^{2+\alpha}}, \end{aligned} \tag{6.1.2}$$

F defined by (4.15). Here

$$\Psi_{n,k} = \tilde{\Psi}_{n,k} + O(k/n), \quad \tilde{\Psi}_{n,k} := \beta \frac{k^2}{n} + \bigl(\beta_1 + O(k/n)\bigr)\frac{k^3}{n^2},$$

with $\beta_1 < 0$, and bounded away from 0, and $\beta < 0$, of order $c_\alpha - c$. So, using

$$1 - e^{-z} \leqslant z, \quad z \geqslant 0,$$

we have: uniformly for $k \leqslant k_n$,

$$\begin{aligned}
\bigl|e^{\Psi_{n,k}} - 1\bigr| &\leqslant \bigl|e^{\tilde{\Psi}_{n,k}}\bigl(e^{O(k/n)} - 1\bigr)\bigr| + \bigl|e^{\tilde{\Psi}_{n,k}} - 1\bigr| \\
&\leqslant \bigl|e^{O(k/n)} - 1\bigr| + |\tilde{\Psi}_{n,k}| \\
&= O\bigl(k/n + (c_\alpha - c)k^2/n + k^3/n^2\bigr).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{1\leqslant k\leqslant k_n} \bigl(x(c)^{k-1}\bigl[x^{k-1}\bigr]F^\alpha(x)\bigr)e^{\Psi_{n,k}} \\
&\quad = \Sigma_0 + O\left(R_n + \frac{\Sigma_0}{n} + \frac{\Sigma_1}{n} + \frac{(c_\alpha - c)\Sigma_2}{n} + \frac{\Sigma_3}{n^2}\right),
\end{aligned} \tag{6.1.3}$$

where

$$\Sigma_t := \sum_{j\geqslant 0} j^t \bigl(x(c)^j\bigl[x^j\bigr]F^\alpha(x)\bigr) = \left(x\frac{d}{dx}\right)^t F^\alpha(x)\Big|_{x=x(c)},$$

and

$$R_n := \sum_{k>k_n} x(c)^{k-1}\bigl[x^{k-1}\bigr]F^\alpha(x) = \alpha \sum_{k>k_n} x(c)^{k-1}\frac{(k(\alpha+1))_{k-2}}{(k-1)!}.$$

Now, by the definition of $x(c)$ and (4.22) with $c < c_\alpha$,

$$F\bigl(x(c)\bigr) = 1 + \frac{c}{\alpha}.$$

Since, using (4.15),

$$\frac{dF}{dx} = \frac{F^{\alpha+2}}{1 - x(\alpha+2)F^{\alpha+1}} = \frac{F^{\alpha+3}}{(\alpha+2) - (\alpha+1)F}, \tag{6.1.4}$$

we then have

$$\frac{dF}{dx}\Big|_{x=x(c)} = \frac{F^{\alpha+3}(x(c))}{1 - (\alpha+1)c/\alpha} = O\bigl((c_\alpha - c)^{-1}\bigr),$$

and likewise

$$\left.\frac{d^2F}{dx^2}\right|_{x=x(c)} = O\big((c_\alpha - c)^{-3}\big), \qquad \left.\frac{d^3F}{dx^3}\right|_{x=x(c)} = O\big((c_\alpha - c)^{-5}\big).$$

Consequently

$$\begin{aligned} \Sigma_0 &= F^\alpha\big(x(c)\big) = \left(1 + \frac{c}{\alpha}\right)^\alpha, \\ \Sigma_t &= O\big((c_\alpha - c)^{-2t+1}\big), \quad t = 1, 2, 3, \end{aligned} \tag{6.1.5}$$

whence, as $n(c_\alpha - c)^3 \to \infty$,

$$\frac{\Sigma_0}{n} + \frac{\Sigma_1}{n} + \frac{(c_\alpha - c)\Sigma_2}{n} + \frac{\Sigma_3}{n^2} = O\big(n^{-1}(c_\alpha - c)^{-2}\big), \tag{6.1.6}$$

the dominant contribution coming from the Σ_2 term.

Turn to R_n. Writing

$$\frac{(k(\alpha+1))_{k-2}}{(k-1)!} = \frac{\Gamma(k(\alpha+2)-2)}{\Gamma(k(\alpha+1))\Gamma(k)},$$

and applying the Stirling formula for the Gamma function, we see that R_n is of order

$$\begin{aligned} \sum_{j \geqslant k_n} j^{-3/2}\left[\frac{(\alpha+2)^{\alpha+2}}{(\alpha+1)^{\alpha+1}} \cdot \frac{c\alpha^{\alpha+1}}{(\alpha+c)^{2+\alpha}}\right]^j &= \sum_{j \geqslant k_n} j^{-3/2}\big(x^{-1}(c_\alpha)x(c)\big)^j \\ &\leqslant \frac{(x(c)/x(c_\alpha))^{k_n}}{k_n^{3/2}(1 - x(c)/x(c_\alpha))}, \end{aligned} \tag{6.1.7}$$

as

$$x(c) < x(c_\alpha), \quad \forall c \neq c_\alpha.$$

More precisely, since $x''(c_\alpha) < 0$,

$$\frac{x(c)}{x(c_\alpha)} \leqslant 1 - \sigma(c_\alpha - c)^2, \quad \sigma > 0, \tag{6.1.8}$$

and we then have

$$R_n = O\left(\frac{e^{-\sigma_1 k_n (c_\alpha - c)^2}}{k_n^{3/2}(c_\alpha - c)^2}\right), \quad \sigma_1 > 0. \tag{6.1.9}$$

Putting (6.1.2)–(6.1.9) together, and using the definition (6.1.1) of k_n, we obtain

$$\begin{aligned} S_n &= n + O\big((c_\alpha - c)^{-2}\big) + O\left(\frac{ne^{-\sigma_1 k_n (c_\alpha - c)^2}}{k_n^{3/2}(c_\alpha - c)^2}\right) \\ &= n + O\big((c_\alpha - c)^{-2}\big) + O\big(\omega^{-\sigma_1 A + 1}(c_\alpha - c)^{-2}\big) \\ &= n + O\big((c_\alpha - c)^{-2}\big), \end{aligned} \tag{6.1.10}$$

for $A > \sigma_1^{-1}$. Therefore the expected number of vertices left after deletion of all tree components of size k_n, at most, is of order $O((c_\alpha - c)^{-2})$, hence dwarfed by k_n. Thus

$$\lim_{n\to\infty} \mathbf{Pr}\left\{L_n \leqslant \lambda \frac{\ln[n(c_\alpha - c)^3]}{(c - c_\alpha)^2}\right\} = 1, \quad \lambda > A.$$

To complete the proof of Theorem 1(a), we need to show that, for some small $\varepsilon > 0$, $\hat{X}$, the number of tree components of size in $[k_1, k_2]$,

$$k_1 = \left\lceil \frac{\varepsilon^2 \ln[n(c_\alpha - c)^3]}{(c - c_\alpha)^2} \right\rceil, \qquad k_2 = \left\lceil \frac{\varepsilon \ln[n(c_\alpha - c)^3]}{(c - c_\alpha)^2} \right\rceil,$$

tends to infinity in probability. Since $(c_\alpha - c)k_i^2/n \to 0$, $k_i^3/n^2 \to 0$, it follows from Lemma 4 ((4.3)–(4.4)) that

$$\mathbf{E}\big[\hat{X}(\hat{X} - 1)\big] = \sum_{i,j \in [k_1,k_2]} E_{i,j} \sim \left(\sum_{i \in [k_1,k_2]} E_i\right)^2 = \big(\mathbf{E}[\hat{X}]\big)^2, \tag{6.1.11}$$

where

$$E_i \sim n \frac{\alpha^i c^{i-1}}{(\alpha + c)^{2(i-1)}} \left(\frac{\alpha}{\alpha + c}\right)^{i\alpha} \cdot \frac{(i(\alpha + 1))_{i-2}}{i!}.$$

That is, using $x(c)$ defined in (4.17) and applying the Stirling formula, as in the estimation (6.1.7) of R_n,

$$\begin{aligned} E_i &\sim n\alpha \left(\frac{\alpha}{\alpha + c}\right)^\alpha x(c)^{i-1} \frac{(i(\alpha + 1))_{i-2}}{i!} \\ &= n\alpha \left(\frac{\alpha}{\alpha + c}\right)^\alpha x(c)^{i-1} \frac{\Gamma(i(\alpha + 1) + i - 2)}{\Gamma(i + 1)\Gamma(i(\alpha + 1))} \\ &\sim g(c) \frac{n}{i^{5/2}} \left(\frac{x(c)}{x(c_\alpha)}\right)^i; \quad g(c) := \alpha \left(\frac{\alpha}{\alpha + c}\right)^\alpha \sqrt{\frac{\alpha + 1}{2\pi(\alpha + 2)^5}} x(c)^{-1}. \end{aligned} \tag{6.1.12}$$

Observe that $1 - x(c)/x(c_\alpha)$ is exactly of order $(c - c_\alpha)^2$. Using (6.1.12), we see that $E[\hat{X}]$ is at least of order

$$n\left(\frac{x(c)}{x(c_\alpha)}\right)^{k_2}\sum_{i=k_1}^{k_2} i^{-5/2} \geqslant \frac{1}{3}nk_1^{-3/2}\left(\frac{x(c)}{x(c_\alpha)}\right)^{k_2}$$
$$\geqslant \frac{[n(c_\alpha - c)^3]^{1-\sigma\varepsilon}}{3(\ln[n(c_\alpha - c)^3])^{3/2}} \to \infty,$$

if $\varepsilon > 0$ is sufficiently small. Therefore, by (6.1.11),

$$\mathbf{E}\big[\hat{X}^2\big] = \mathbf{E}\big[\hat{X}(\hat{X}-1)\big] + \mathbf{E}[\hat{X}] \sim \mathbf{E}^2[\hat{X}],$$

and (Chebyshev's inequality) $\hat{X} \to \infty$ in probability.

6.2. *Proof of Theorem 1(b)*

To bound L_n from above, it suffices to consider c satisfying the condition

$$c - c_\alpha = \delta n^{-1/3},$$

where $\delta > 0$ is fixed. Define $k_n = [(c - c_\alpha)^{-2}]$; in particular, $k_n^4/n^3 = o(1)$. By Lemma 4 (4.3), uniformly for $k \leqslant k_n$,

$$\begin{aligned}
\Psi_{n,k} &= \Psi^*_{n,k} + O(k/n),\\
\Psi^*_{n,k} &:= \beta(c)\frac{k^2}{n} + \beta_1(c)\frac{k^3}{n^2} = \sigma_n\frac{k^2}{n^{4/3}} + \beta_1(c)\frac{k^3}{n^2},\\
\sigma_n &:= \frac{1}{2}\left(\frac{\alpha+1}{\alpha}\right)^2\delta + O\big(n^{-1/3}\big).
\end{aligned}$$

Here

$$\Psi^*_{n,k}\ \begin{cases} \geqslant 0, & k \leqslant k_n^* := \frac{\sigma_n}{-\beta_1(c)}n^{2/3},\\ < 0, & k > k_n^*. \end{cases}$$

Importantly, the switch point k_n^* depends on δ only. So, for $k > k_n^*$, we have

$$\big|e^{\Psi^*_{n,k}} - 1\big| \leqslant \big|\Psi^*_{n,k}\big|.$$

For $k \leqslant k_n^*$,

$$0 \leqslant \Psi^*_{n,k} \leqslant \sigma_n\frac{(k_n^*)^2}{n^{4/3}} \leqslant B,$$

where a constant B depends on δ only, too. Thus

$$\big|e^{\Psi^*_{n,k}} - 1\big| = O\big(\big|\Psi^*_{n,k}\big|\big),$$

uniformly for all k. So, uniformly for $k \leqslant k_n$,

$$\begin{aligned}\left|e^{\Psi_{n,k}}-1\right| &= O\left(\left|e^{\Psi^*_{n,k}}-1\right|\right)+O(k/n)=O\left(\left|\Psi^*_{n,k}\right|\right)+O(k/n)\\ &= O\left(k/n+(c-c_\alpha)k^2/n+k^3/n^2\right).\end{aligned}$$

Consequently, only minor modifications of the proof of (6.1.10) are needed to show the following: for some $\sigma_1>0$,

$$\begin{aligned}S_n &= n+O\left((c_\alpha-c)^{-2}\right)+O\left(\frac{ne^{-\sigma_1 k_n(c_\alpha-c)^2}}{k_n^{3/2}(c_\alpha-c)^2}\right)\\ &= n+O\left((c_\alpha-c)^{-2}\right).\end{aligned}\tag{6.2.1}$$

Then

$$n-S_n=O\left((c-c_\alpha)^{-2}\right)=O\left(n^{2/3}\right),\tag{6.2.2}$$

and we conclude that, for $\lambda=\lambda(n)\to\infty$ however slowly,

$$\lim_{n\to\infty}\mathbf{Pr}\left\{L_n\leqslant\frac{\lambda}{(c-c_\alpha)^2}\right\}=1.$$

Let us bound $L_{n,t}$ from below. Introduce

$$\bar{k}=\left\lfloor\varepsilon_n n^{2/3}\right\rfloor,$$

where $\varepsilon_n\downarrow 0$ however slowly. Then in (6.1.2)

$$\Psi_{n,k}=O\left(k/n+|c-c_\alpha|k^2/n\right)+O\left(k^3/n^2\right)=O(\varepsilon_n),\quad k\in[\bar{k},2\bar{k}].$$

So, as in (6.1.12), uniformly for $k\in[\bar{k},2\bar{k}]$ we have

$$E_k\sim g(c)\frac{n}{k^{5/2}}\left(\frac{x(c)}{x(c_\alpha)}\right)^k.$$

This time though

$$\left(\frac{x(c)}{x(c_\alpha)}\right)^k=\left(1+O\left((c-c_\alpha)^2\right)\right)^k=1+O(\varepsilon_n),\quad k\in[\bar{k},2\bar{k}].$$

Consequently, for $\bar{X}$, the number of tree components of size from $[\bar{k},2\bar{k}]$, we have

$$\sum_{k=\bar{k}}^{2\bar{k}}E_k\geqslant\sigma\frac{n}{\bar{k}^{3/2}}\geqslant\varepsilon_n^{-3/2}\to\infty.$$

So, arguing as in the case of $\hat{X}$ in Section 6.1, we obtain that $\bar{X}\to\infty$, in probability. Therefore, for each fixed $t\geqslant 1$

$$\mathbf{Pr}\{L_{n,t}\geqslant\bar{k}\}\to 1.$$

In the next section we will consider a supercritical case, when $c > c_\alpha$ and $c - c_\alpha$ is not too small. To do so, we need a deeper look at the nearcritical case. Consider a random multigraph $MG_\alpha(N, \mathcal{M})$ with

$$c - c_\alpha = \frac{2\mathcal{M}}{N} - c_\alpha = \Theta\big(N^{-1/3}\big).$$

Introduce

$$D_N = N - \sum_{k \leqslant k_N} kX_k,$$

where X_k is the count of tree components of size k in $MG_\alpha(N, \mathcal{M})$, and $k_N = [(c - c_\alpha)^{-2}]$. What we showed while proving Theorem 1(b) (see (6.2.1)–(6.2.2)) was that

$$\mathbf{E}[D_N] \leqslant_b N^{2/3}. \tag{6.2.3}$$

Let us show that

$$\mathbf{E}\big[D_N^2\big] \leqslant_b N^{4/3}, \tag{6.2.4}$$

as well.

Recalling that $E_{i,j}$ is the expected number of ordered pairs of distinct trees of sizes i and j, after simple algebra we obtain

$$\mathbf{E}\big[D_N^2\big] = \mathbf{E}^2[D_N] + \sum_{i,j \leqslant k_N} ij(E_{i,j} - E_i E_j) + \sum_{i \leqslant k_N} i^2 E_i. \tag{6.2.5}$$

By (6.2.3), the first summand is $O(N^{4/3})$. The third summand is of order $N\Sigma_1$, see (6.1.2), (6.1.4), and $\Sigma_1 = O((c - c_\alpha)^{-1})$, see (6.1.5), i.e.

$$\sum_{i \leqslant k_n} i^2 E_i \leqslant_b N(c - c_\alpha)^{-1} \leqslant_b N^{4/3}. \tag{6.2.6}$$

Further, by Lemma 4 (4.4) and $\beta(c) = O(N^{-1/3})$,

$$|E_{i,j} - E_i E_j| \leqslant_b N^{-4/3} ij E_i E_j + N^{-1}(i + j) E_i E_j. \tag{6.2.7}$$

So the middle term in (6.2.5) is of order

$$\begin{aligned} &N^{-4/3}\Big(\sum_{i \leqslant k_N} i^2 E_i\Big)^2 + N^{-1} \sum_{i \leqslant k_N} i^2 E_i \cdot \sum_{j \leqslant k_N} jE_j \\ &\quad \leqslant_b N^{-4/3}\big(N^{4/3}\big)^2 + N^{-1} N^{4/3} N \leqslant_b N^{4/3}, \end{aligned}$$

and (6.2.3) follows.

Lemma 6. *Let $V_1, V_2, \dots$ denote the vertex sets of the components of $MG_\alpha(N, \mathcal{M})$. If $2\mathcal{M}/N - c_\alpha = \Theta(N^{-1/3})$, then*

$$\mathbf{E}\Big[\sum_j |V_j|^2\Big] \leqslant_b N^{4/3}.$$

Proof. Combine an obvious bound

$$\mathbf{E}\Big[\sum_j |V_j|^2\Big] \leqslant \sum_{i \leqslant k_N} i^2 \mathbf{E}[X_i] + \mathbf{E}\big[D_N^2\big],$$

with (6.2.4) and (6.2.6). □

Now, whatever the values $N > k$ and $\mathcal{M} > \ell$ are, we can replace (5.4) with

$$\begin{aligned} E_{k,\ell}(N, \mathcal{M}) &= \binom{N}{k}\binom{\mathcal{M}}{\ell} C_\alpha(k,\ell) \frac{((N-k)\alpha)_{2(\mathcal{M}-\ell)}}{(N\alpha)_{2\mathcal{M}}} \\ &\leqslant \frac{1}{k^2}\mathbf{E}\Big[\sum_j |V_j|^2\Big]. \end{aligned} \tag{6.2.8}$$

Recall that in the proof of Lemma 5 we set $N = \lceil k^{3/2}/t^{1/6}\rceil$, or $N = \lceil k^{3/2}\rceil$, and $\mathcal{M} = \lceil Nc_\alpha/2\rceil$. To fit the condition of Lemma 6, let us redefine $\mathcal{M}$ a bit, setting $\mathcal{M} = \lceil N(c_\alpha + N^{-1/3})/2\rceil$. By Lemma 6, the right-hand side in (6.2.8) is of order

$$\frac{N^{4/3}}{k^2} \leqslant_b \frac{(k^{3/2})^{4/3}}{k^2} = 1,$$

instead of the initial N/k in (5.4). Aside from N/k being now replaced with 1, the modification of $\mathcal{M}$ does not change the rest of the proof of Lemma 5. Hence we have strengthened the bound (5.6)–(5.7) in Lemma 5.

Lemma 7. *Let $k \leqslant Bn^{2/3}$, $B > 0$ being fixed. There exists a constant $A = A(B) > 0$ such that, uniformly for $t := \ell - k \in (0, k]$,*

$$E_{k,\ell}(n,m) \leqslant_b \left(\frac{A}{t^{1/6}}\frac{k^{3/2}}{n}\right)^t \left(\frac{x(c)}{x(c_\alpha)}\right)^k e^{\beta(c)k^2/n}. \tag{6.2.9}$$

Furthermore,

$$E_{k,k}(n,m) \leqslant_b \left(\frac{x(c)}{x(c_\alpha)}\right)^k e^{\beta(c)k^2/n}. \tag{6.2.10}$$

Note. We conjecture that in fact

$$E_{k,\ell}(n,m) \leqslant_b \frac{1}{k}\left(\frac{A}{t^{1/6}}\frac{k^{3/2}}{n}\right)^t \left(\frac{x(c)}{x(c_\alpha)}\right)^k e^{\beta(c)k^2/n},$$

if $k \leqslant Bn^{2/3}$ and $t = \ell - k \in (0, k]$. With this additional factor $1/k$ in place, we would have been able to show that the supercritical phase begins when $n^{1/3}(c - c_\alpha) \to \infty$.

Now we are prepared to deal with the supercritical phase.

6.3. Proof of Theorem 1(c)

So here $n^{1/4}(c - c_\alpha) \to \infty$. We restrict our attention to the case $c - c_\alpha \to 0$, since the argument for $c - c_\alpha$ bounded away from zero is far simpler. Call a component of $MG_\alpha(n, m)$ small or large, if its size is $n^{2/3}$ at most, or strictly above $n^{2/3}$.

Step 1. Let us prove that the total size of small cyclic components is $o_p(n(c - c_\alpha))$, and that quite surely there are no cyclic components of size in $[n^{2/3}/2, n^{2/3}]$. Here and later we use the term "quite surely" (q.s.) from Knuth et al. [21] to indicate that an event in question has probability $1 - O(n^{-K})$, for any $K > 0$. (Not that we actually need those probabilities to be that close to 1; a lower bound $1 - O(n^{-2})$, say, would suffice.)

First, let us show that whp there are no small (k, ℓ)-components with $\ell \geqslant 2k$. Rather crudely, by (2.15),

$$E_{k,\ell}(n, m) \leqslant \binom{n}{k} P(k, \ell), \qquad P(k, \ell) = \binom{m}{\ell} \cdot \frac{(k\alpha)_{2\ell}((n - k)\alpha)_{2(m-\ell)}}{(n\alpha)_{2m}}.$$

Here, for $\ell \geqslant 2k$,

$$\frac{P(k, \ell + 1)}{P(k, \ell)} \leqslant_b \frac{(m - \ell)k^2}{\ell n^2} \leqslant_b \frac{k}{n} \leqslant n^{-1/3}.$$

Further, using (4.10)–(4.11), we have: for some constant $\sigma > 0$,

$$\begin{aligned}\binom{n}{k} P(k, 2k) &= \frac{\langle n\rangle_k \langle m\rangle_{2k}((n - k)a)_{2(m-2k)}}{(n\alpha)_{2m}} \cdot \frac{(k\alpha)_{4k}}{k!(2k)!} \\ &\leqslant \frac{\sigma^k}{n^k} \cdot \frac{k^{4k}}{k^k k^{2k}} \\ &= \left(\frac{\sigma k}{n}\right)^k.\end{aligned}$$

Therefore

$$\sum_{k \leqslant n^{2/3},\, \ell \geqslant 2k} E_{k,\ell}(n, m) \leqslant_b \sum_{k \leqslant n^{2/3}} \binom{n}{k} P(k, 2k) \leqslant_b n^{-1},$$

and so, with probability $1 - O(n^{-1})$, there are no cyclic components in question.

To bound the expected total size of the small cyclic components with $\ell \in (k, 2k]$, we apply Lemma 7 (6.2.9) and obtain

$$\sum_{\substack{1\leqslant k\leqslant n^{2/3}\\ k<\ell\leqslant 2k}} kE_{k,\ell}(n,m) \leqslant_b \sum_{1\leqslant k\leqslant n^{2/3}} k\left(\frac{x(c)}{x(c_\alpha)}\right)^k e^{\beta(c)k^2/n}\sum_{t>0}\left(\frac{A}{t^{1/6}}\frac{k^{3/2}}{n}\right)^t$$
$$\leqslant_b \frac{1}{n}\sum_{1\leqslant k\leqslant n^{2/3}} k^{5/2}\left(\frac{x(c)}{x(c_\alpha)}\right)^k e^{\beta(c)k^2/n}. \tag{6.3.1}$$

Here, as $c-c_\alpha \gg n^{-1/3}$,

$$\left(\frac{x(c)}{x(c_\alpha)}\right)^k e^{\beta(c)k^2/n} \leqslant_b \exp\bigl[-\sigma k(c-c_\alpha)^2\bigr],$$

and a resulting function $h(k) := (5/2)\ln k - \sigma k(c-c_\alpha)^2$ attains its maximum at

$$k_{\max} = \frac{5}{2\sigma}(c-c_\alpha)^{-2}.$$

Since $h''(k) = -(5/2)k^{-2}$, a standard argument shows then that

$$\sum_{\substack{1\leqslant k\leqslant n^{2/3}\\ k<\ell\leqslant 2k}} kE_{k\ell}(n,m) \leqslant_b n^{-1}\frac{e^{h(k_{\max})}}{\sqrt{-h''(k_{\max})}} \leqslant_b n^{-1}k_{\max}^{7/2} = \sigma_1 n^{-1}(c-c_\alpha)^{-7}, \tag{6.3.2}$$

which is $o(n(c-c_\alpha))$ as $n^{1/4}(c-c_\alpha)\to\infty$. (That's the reason for the exponent $1/4$ in our definition of the supercritical phase!) In addition, as in (6.3.1),

$$\sum_{\substack{n^{2/3}/2\leqslant k\leqslant n^{2/3}\\ k<\ell\leqslant 2k}} E_{k,\ell}(n,m) \leqslant_b \frac{1}{n}\sum_{n^{2/3}/2\leqslant k\leqslant n^{2/3}} k^{3/2}\left(\frac{x(c)}{x(c_\alpha)}\right)^k e^{\beta(c)k^2/n}$$
$$\leqslant_b \frac{1}{n}\sum_{n^{2/3}/2\leqslant k\leqslant n^{2/3}} k^{3/2}\exp\bigl[-\sigma k(c-c_\alpha)^2\bigr]$$
$$\leqslant_b \exp\bigl[-0.4\sigma n^{2/3}(c-c_\alpha)^2\bigr] \leqslant e^{-n^{1/6}}. \tag{6.3.3}$$

Turn now to the expected total size of all small unicyclic components. Using the second and the third lines in (4.24) and (4.26) respectively, and denoting

$$\hat{x} = x(c)\exp\bigl(\beta(c)n^{-1/3}\bigr),$$

we have

$$\sum_{k\leqslant n^{2/3}} kE_{k,k}(n,m) \leqslant_b \sum_{k\geqslant 1} k\hat{x}^{k-1}\bigl[x^k\bigr]\left(\ln\frac{1}{1-z}\right)\Big|_{z=(\alpha+1)(F(x)-1)}$$
$$= \frac{d}{dx}\ln\left(\frac{1}{1-(\alpha+1)(F(x)-1)}\right)\Big|_{x=\hat{x}}$$
$$= \frac{(\alpha+1)F'(\hat{x})}{1-(\alpha+1)(F(\hat{x})-1)}.$$

Since

$$\frac{\beta(c)n^{-1/3}}{(c-c_\alpha)^2} = \frac{1}{n^{1/3}(c-c_\alpha)} \to 0,$$

it follows that $\hat{x} = x(\hat{c})$, where

$$\hat{c} - c_\alpha = (c-c_\alpha)\big(1 + O\big(\beta(c)n^{-1/3}(c-c_\alpha)^{-2}\big)\big) = (c-c_\alpha)\big(1+o(1)\big).$$

Then, by (4.22),

$$\begin{aligned} 1-(\alpha+1)\big(F(\hat{x})-1\big) &= 1-(\alpha+1)\frac{c^*(\hat{c})}{\alpha} = 1 - \frac{c^*(\hat{c})}{c_\alpha} \\ &= \Theta(\hat{c}-c_\alpha) = \Theta(c-c_\alpha), \end{aligned}$$

and, by (6.1.5),

$$\begin{aligned} F'(\hat{x}) &\leqslant_b \big[(\alpha+2)-(\alpha+1)F(\hat{x})\big]^{-1} \\ &= \big[(\alpha+2)-(\alpha+1)\big(1+c^*(\hat{c})/\alpha\big)\big]^{-1} \\ &\leqslant_b (c-c_\alpha)^{-1}. \end{aligned}$$

Therefore

$$\sum_{k\leqslant n^{2/3}} kE_{k,k}(n,m) \leqslant_b (c-c_\alpha)^{-2}, \tag{6.3.4}$$

which is $o(n(c-c_\alpha))$, as $n^{1/3}(c-c_\alpha)\to\infty$. And, using Lemma 7 (6.2.10), analogously to (6.3.3) we obtain

$$\sum_{n^{2/3}/2\leqslant k\leqslant n^{2/3}} E_{k,k}(n,m) \leqslant e^{-n^{1/6}}. \tag{6.3.5}$$

Let C_n stand for the total size of all small cyclic components. From (6.3.2) and (6.3.4), it follows that

$$\mathbf{E}[C_n] \leqslant_b n^{-1}(c-c_\alpha)^{-7} + (c-c_\alpha)^{-2}. \tag{6.3.6}$$

The relations (6.3.3) and (6.3.5) together imply that q.s. $MG_\alpha(n,m)$ has no cyclic components of size in $[0.5n^{2/3}, n^{2/3}]$. Consequently, q.s. in the graph process $\{MG_\alpha(n,\mu)\}_{\mu\leqslant m}$ there is never a moment when a current multigraph has a cyclic component of size in question.

Step 2. Introduce Y_n and $\mathcal{Y}_n$, the total size of small tree components and the total size of even smaller tree components, each with at most

$$k_n := \left\lceil \frac{A\ln[n|c-c_\alpha|^3]}{(c-c_\alpha)^2} \right\rceil$$

vertices. Let us prove that whp $Y_n = \mathcal{Y}_n$, and that the distribution of $\mathcal{Y}_n$, whence of Y_n itself, is sharply concentrated around

$$\mathbf{E}[\mathcal{Y}_n] = \sum_{k \leqslant k_n} kE_k,$$

if A is sufficiently large. In the part (a) of the proof, we already evaluated asymptotically $E[\mathcal{Y}_n]$ for $n^{1/3}(c - c_\alpha) \to -\infty$. The same argument, without any changes, works in the case $n^{1/3}(c - c_\alpha) \to \infty$. The only difference is that the formula (6.1.4) now becomes

$$\Sigma_0 = F^\alpha\big(x(c)\big) = \left(1 + \frac{c^*}{\alpha}\right)^\alpha,$$

where $c^* < c_\alpha$ is defined in the second line of (4.22). Consequently, cf. (6.1.10),

$$\begin{aligned}\mathbf{E}[\mathcal{Y}_n] &= n\left(\frac{\alpha}{\alpha + c}\right)^\alpha \Sigma_0 + O\big((c - c_\alpha)^{-2}\big)\\ &= n\left(\frac{c^* + \alpha}{c + \alpha}\right)^\alpha + O\big((c - c_\alpha)^{-2}\big).\end{aligned} \tag{6.3.7}$$

Further, since the estimates (6.1.7)–(6.1.9) hold for $c > c_\alpha$ as well,

$$\begin{aligned}\sum_{k_n \leqslant k \leqslant n^{2/3}} kE_k &\leqslant_b \frac{ne^{-\sigma_1 k_n (c-c_\alpha)^2}}{k_n^{3/2}(c - c_\alpha)^2}\\ &\leqslant_b (c - c_\alpha)^{-2}\big[n(c - c_\alpha)^3\big]^{1-\sigma_1 A} \ll (c - c_\alpha)^{-2},\end{aligned} \tag{6.3.8}$$

provided that $A > 1/\sigma_1$. Therefore

$$\mathbf{E}[Y_n - \mathcal{Y}_n] = o\big((c - c_\alpha)^{-2}\big).$$

Since each small tree component counted in $Y_n - \mathcal{Y}_n$ has $k_n \gg (c - c_\alpha)^{-2}$ vertices at least, the last estimate implies that whp $Y_n = \mathcal{Y}_n$, i.e. whp there are no small tree components of size k_n at least. Besides, analogously to (6.3.8),

$$\sum_{n^{2/3}/2 \leqslant k \leqslant n^{2/3}} kE_k \leqslant_b \frac{ne^{-\sigma_1 k (c-c_\alpha)^2}}{k^{3/2}(c - c_\alpha)^2}\bigg|_{k=n^{2/3}/2} \leqslant e^{-n^{1/6}}.$$

Thus q.s. neither $MG_\alpha(n, m)$ nor $MG_\alpha(n, \mu)$, $\mu < m$, have a tree component of size in $[0.5n^{2/3}, n^{2/3}]$.

To show that $\mathcal{Y}_n$ is sharply concentrated around $\mathbf{E}[\mathcal{Y}_n]$, we need to use Lemma 4(c) again. Since

$$\frac{(c - c_\alpha)k_n^2}{n} \to 0, \quad \frac{k_n}{n} \to 0,$$

by (4.4) we have: for k_1, $k_2 \leqslant k_n$,

$$|E_{k_1,k_2} - E_{k_1}E_{k_2}| \leqslant_b \frac{c-c_\alpha}{n} k_1k_2E_{k_1}E_{k_2} + \frac{1}{n}(k_1+k_2)E_{k_1}E_{k_2} + \frac{1}{n^2}(k_1+k_2)k_1k_2E_{k_1}E_{k_2}.$$

So, analogously to the proof of (6.2.4),

$$\begin{aligned}\mathbf{E}\big[\mathcal{Y}_n(\mathcal{Y}_n-1)\big] &= \sum_{k_1,k_2\leqslant k_n} k_1k_2E_{k_1,k_2} + \sum_{k\leqslant k_n} k(k-1)E_k \\ &\leqslant_b \mathbf{E}^2[\mathcal{Y}_n] + \frac{(c-c_\alpha)}{n}\Big(\sum_{k\leqslant k_n} k^2E_k\Big)^2 + \frac{1}{n}\mathbf{E}[\mathcal{Y}_n]\sum_{k\leqslant k_n} k^2E_k \\ &\quad + \frac{1}{n^2}\sum_{k\leqslant k_n} k^2E_k \cdot \sum_{k\leqslant k_n} k^3E_k + \sum_{k\leqslant k_n} k^2E_k \\ &= \mathbf{E}^2[\mathcal{Y}_n] + O\big(n(c-c_\alpha)^{-1}\big),\end{aligned}$$

as

$$\sum_{k\leqslant k_n} k^rE_k = O\big(n(c-c_\alpha)^{-2r+3}\big), \quad r \geqslant 2,$$

for $n(c-c_\alpha)^3 \to \infty$. Thus

$$\begin{aligned}\mathbf{Var}[\mathcal{Y}_n] &= \mathbf{E}\big[\mathcal{Y}_n(\mathcal{Y}_n-1)\big] + \mathbf{E}[\mathcal{Y}_n] - \mathbf{E}^2[\mathcal{Y}_n] \\ &= O(n) + O\big(n(c-c_\alpha)^{-1}\big) = O\big(n(c-c_\alpha)^{-1}\big).\end{aligned}$$

So, by Chebyshev's inequality,

$$\mathcal{Y}_n = \mathbf{E}[\mathcal{Y}_n] + O_p\big(\sqrt{n(c-c_\alpha)^{-1}}\big). \tag{6.3.9}$$

Using (6.3.7), $Y_n = \mathcal{Y}_n$ whp, and $n^{1/3}(c-c_\alpha) \to \infty$, we can replace (6.3.9) with

$$Y_n = n\left(\frac{c^*+\alpha}{c+\alpha}\right)^\alpha + O_p\big(\sqrt{n(c-c_\alpha)^{-1}}\big). \tag{6.3.10}$$

Combining (6.3.6) *and* (6.3.10)*, and using* $\sqrt{n(c-c_\alpha)^{-1}} \gg (c-c_\alpha)^{-2}$*, we conclude that*

$$Y_n + C_n = n\left(\frac{c^*+\alpha}{c+\alpha}\right)^\alpha + O_p\big(n^{-1}(c-c_\alpha)^{-7} + \sqrt{n(c-c_\alpha)^{-1}}\big). \tag{6.3.11}$$

Also, q.s. the multigraphs $MG_\alpha(n,\mu)$ $(\mu \leqslant m)$*, have no components of size in* $[0.5n^{2/3}, n^{2/3}]$.

Step 3. Let us prove that whp $MG_\alpha(n,m)$ has exactly one large component.

Pick $\gamma > 0$ and set

$$\hat{c} = c - \gamma n^{-1/3};$$

clearly

$$n^{1/4}(\hat{c} - c_\alpha) = n^{1/4}(c - c_\alpha) - \gamma n^{-1/12} \to \infty, \quad \frac{\hat{c} - c_\alpha}{c - c_\alpha} \to 1.$$

We consider $MG_\alpha(n,m)$ and $MG_\alpha(n,\hat{m})$, $\hat{m} = \hat{c}n/2$, as the two states in the multigraph growth process $\{MG_\alpha(n,\mu)\}_{\mu \geqslant 0}$. What we proved in Steps 1 and 2 means that q.s. no new large component has been born during the time interval $[n\hat{c}/2, m]$: if it has, then at its birth a larger of two merging small components would have had size between $n^{2/3}/2$ and $n^{2/3}$.

Further, introducing $\hat{Y}_n$, $\hat{C}_n$, the corresponding parameters for $MG_\alpha(n,\hat{m})$, we have

$$n - (\hat{X}_n + \hat{Y}_n) = n\left[1 - \left(\frac{\hat{c}^* + \alpha}{\hat{c} + \alpha}\right)^\alpha\right] + O_p\left(n^{-1}(c - c_\alpha)^{-7} + \sqrt{n(c - c_\alpha)^{-1}}\right).$$

The explicit term here is of an exact order $n(\hat{c} - c_\alpha) \sim n(c - c_\alpha)$, thus dwarfing the remainder term. That is, whp $n - (\hat{X}_n + \hat{Y}_n)$ is of the exact order $n(\hat{c} - c_\alpha) \gg n^{2/3}$. So, introducing $\mathcal{S}_n = \mathcal{S}(MG_\alpha(n,\hat{m}))$, the total size of the large components of $MG_\alpha(n,\hat{m})$, we see that $\mathcal{S}_n/n^{2/3} \to \infty$, in probability. Let A_n denote the event that $MG_\alpha(n,\hat{m})$ has large components and that they all will have merged by time m; so $A_n \subset \{\mathcal{S}_n > 0\}$. Assuming that $\mathcal{S}_n > 0$, let us bound $\mathbf{Pr}(A_n^c \mid MG_\alpha(n,\hat{m}))$.

To this end, let us partition a vertex set of each large component of $MG_\alpha(n,\hat{m})$ into subsets of cardinality from $\lfloor 0.4n^{2/3} \rfloor$ to $\lfloor n^{2/3} \rfloor$. (Such a partition is obtained via an obvious algorithm that determines it one subset at a time.) Let $V_1, \ldots, V_\nu$ be the collection of all the subsets from all the partitions. Then $\nu \geqslant \frac{\mathcal{S}_n}{n^{2/3}} \to \infty$, in probability. On the event $A_n^c \cap \{\mathcal{S}_n > 0\}$ there exists $B \subset [\nu]$, $1 \leqslant |B| \leqslant \nu/2$, such that during the time interval $[\hat{m}, m]$ no edge will appear between $\bigcup_{i \in B} V_i$ and $\bigcup_{i \in B^c} V_i$. By the definition of the multigraph process, for a given set B, the probability of such an outcome, conditioned on $MG_\alpha(n,\hat{m})$, is bounded above by

$$\begin{aligned}\left(1 - \sigma \frac{|\bigcup_{i \in B} V_i||\bigcup_{i \in B^c} V_i|}{n^2}\right)^{m - \hat{m}} &\leqslant \left(1 - \sigma_1 \frac{|B||B^c|}{n^{2/3}}\right)^{m - \hat{m}} \\ &\leqslant \exp\left(-\sigma_2 |B|\left|B^c\right|\right).\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{Pr}\left(A_n^c \mid MG_\alpha(n,\hat{m})\right) &\leqslant \sum_{1 \leqslant k \leqslant \nu/2} \binom{\nu}{k} \exp\left(-\sigma_2 k(\nu - k)\right) \\ &\leqslant_b \nu e^{-\sigma_2(\nu - 1)} \leqslant_b e^{-\sigma^* \mathcal{S}_n / n^{2/3}},\end{aligned}$$

which tends to zero in probability. So $\mathbf{Pr}(A_n^c) \to 0$, as well.

Thus whp $MG_\alpha(n,m)$ has exactly one large component, of size

$$\begin{aligned} n-(Y_n+C_n) &= n\left[1-\left(\frac{c^*+\alpha}{c+\alpha}\right)^\alpha\right]+O_p\left(n^{-1}(c-c_\alpha)^{-7}+\sqrt{n(c-c_\alpha)^{-1}}\right)\\ &= n\left[1-\left(\frac{c^*+\alpha}{c+\alpha}\right)^\alpha\right]\left(1+O_p\left[\left(n^{1/4}(c-c_\alpha)\right)^{-8}+\left(n^{1/3}(c-c_\alpha)\right)^{-3/2}\right]\right). \end{aligned}$$

Since $n^{1/4}(c-c_\alpha)\to\infty$, this completes the proof of Theorem 1(c).

7. Connectedness threshold

Introduce $m(n)=n^{1+\alpha^{-1}}$.

Theorem 2.

(a)
$$\lim_{n,m\to\infty}\mathbf{Pr}\left\{MG_\alpha(n,m)\ \textit{is connected}\right\}=\begin{cases}0, & \textit{if } \frac{m}{m(n)}\to 0,\\ 1, & \textit{if } \frac{m}{m(n)}\to\infty.\end{cases} \tag{7.1}$$

(b) *If*

$$\lim\left(\frac{\alpha}{2}\right)^\alpha\cdot\frac{n^{1+\alpha}}{m^\alpha}=x\in(0,\infty), \tag{7.2}$$

then

$$\lim_{n,m\to\infty}\mathbf{Pr}\left\{MG_\alpha(n,m)\ \textit{is connected}\right\}=e^{-x}. \tag{7.3}$$

Proof. (1) Let X_n denote the total number of zero-degree vertices. Suppose that (7.2) holds. Let us show that X_n converges, in distribution, to Poisson(x). To this end, we need to show that the factorial moments of X_n converge to those of Poisson(x). By (2.16),

$$\mathbf{E}\left[\langle X_n\rangle_k\right]=\langle n\rangle_k\frac{((n-k)\alpha)_{2m}}{(n\alpha)_{2m}}\sim\left[\frac{n^{1+\alpha}}{m^\alpha}\left(\frac{\alpha}{2}\right)^\alpha\right]^k\to x^k,$$

for each $k\geqslant 1$. (Regarding the estimate of the fraction in $\mathbf{E}[\langle X_n\rangle_k]$ see the proof of Lemma 4 (4.4).) Hence

$$\lim_{n\to\infty}\mathbf{Pr}(X_n=k)=e^{-x}\frac{x^k}{k!},\quad k\geqslant 0,$$

so that

$$\lim_{n\to\infty}\mathbf{Pr}(X_n>0)=1-e^{-x}.$$

Since $1 - e^{-x} \to 1$ as $x \to \infty$, we see that $\mathbf{Pr}(X_n > 0) \to 1$ if $m/m(n) \to 0$. Therefore in this case $MG_\alpha(m, n)$ is disconnected with probability approaching 1. On the other hand,

$$\sum_{k>K} e^{-x}\frac{x^k}{k!} \to 0, \quad K \to \infty,$$

uniformly for $x \geqslant \sigma$, $\sigma > 0$ fixed. Therefore if $\liminf m/m(n) > 0$, then X_n is bounded in probability, $X_n = O_p(1)$.

(2) Suppose that $m/m(n) \in [a, n]$, $a > 0$. Let us show that whp $MG_\alpha(n, m)$ consists of a number of zero degree vertices, with the remaining vertices forming a single component. Let A_n denote the event in question. On A_n^c, $MG_\alpha(n, m)$ must contain a component with two vertices at least, and $m/2$ edges at most. Let C_n denote the number of such components. Then

$$\mathbf{Pr}\big(A_n^c\big) \leqslant \mathbf{Pr}(C_n > 0) \leqslant \mathbf{E}[C_n].$$

By (2.15),

$$\mathbf{E}[C_n] \leqslant \sum_{\substack{2\leqslant k\leqslant n-2\\ k-1\leqslant\mu\leqslant m/2}} \binom{n}{k} P(k, \mu),$$

$$P(k, \mu) = \binom{m}{\mu} \cdot \frac{(k\alpha)_{2\mu}((n-k)\alpha)_{2(m-\mu)}}{(n\alpha)_{2m}}.$$

Step 1. Let us show that, for some constant $\gamma > 0$, the terms of this sum with $\mu \geqslant \gamma n$ can be neglected. First, for $\mu > 0$,

$$\begin{aligned}
\ln\big((k\alpha)_{2\mu}\big) &= \sum_{j=0}^{2\mu-1} \ln(k\alpha + j) \\
&\leqslant \int_0^{2\mu} \ln(k\alpha + x)\,dx \\
&= \big[(k\alpha + x)\ln(k\alpha + x) - x\big]\Big|_0^{2\mu} \\
&= (k\alpha + 2\mu)\ln(k\alpha + 2\mu) - k\alpha\ln(k\alpha) - 2\mu.
\end{aligned}$$

Likewise

$$\begin{aligned}
&\ln\big[\big((n-k)\alpha\big)_{2(m-\mu)}\big] \\
&\quad\leqslant \big((n-k)\alpha + 2(m-\mu)\big)\ln\big[(n-k)\alpha + 2(m-\mu)\big] \\
&\qquad - \alpha(n-k)\ln\big((n-k)\alpha\big) - 2(m-\mu).
\end{aligned}$$

And

$$\ln\big[(n\alpha)_{2m}\big] \geqslant \int_{-1}^{2m-1} \ln(n\alpha + x)\,dx$$
$$= (2m-1+n\alpha)\ln(n\alpha+2m-1) - (n\alpha-1)\ln(n\alpha-1) - 2m$$
$$= (2m+n\alpha)\ln(n\alpha+2m) - n\alpha\ln(n\alpha) - 2m + O(n),$$

as $n \ll m$. As

$$n\ln n - k\ln k - (n-k)\ln(n-k) \leqslant n\ln 2,$$

we obtain then that

$$\frac{(k\alpha)_{2\mu}((n-k)\alpha)_{2(m-\mu)}}{(n\alpha)_{2m}} \leqslant \exp\left[-(2m+n\alpha)H\left(\frac{k\alpha+2\mu}{2m+n\alpha}\right) + O(n)\right];$$
$$H(x) := x\ln\frac{1}{x} + (1-x)\ln\frac{1}{1-x}.$$

Observe that $H(x)$ is concave, and $H(x) = H(1-x)$; consequently, for $\varepsilon \in (0, 1/2)$,

$$\max\big\{\big|H'(x)\big|\colon\ x \in [\varepsilon, 1-\varepsilon]\big\} = H'(\varepsilon). \tag{7.4}$$

Further, for $1 \leqslant k \leqslant n-1$, $\mu \leqslant m/2$,

$$\frac{\alpha}{2m+n\alpha} \leqslant \frac{k\alpha+2\mu}{2m+n\alpha} \leqslant 1 - \frac{\alpha}{2m+n\alpha}. \tag{7.5}$$

So, as $\binom{n}{k} \leqslant 2^n$,

$$\binom{n}{k}P(k,\mu) \leqslant \exp\left[mH\left(\frac{\mu}{m}\right) - (2m+n\alpha)H\left(\frac{k\alpha+2\mu}{2m+n\alpha}\right) + O(n)\right].$$

Here, by concavity of $H(x)$ and (7.4)–(7.5),

$$H\left(\frac{k\alpha+2\mu}{2m+n\alpha}\right) \geqslant H\left(\frac{\mu}{m}\right) - H'\left(\frac{k\alpha+2\mu}{2m+n\alpha}\right)\left(\frac{\mu}{m} - \frac{k\alpha+2\mu}{2m+n\alpha}\right)$$
$$\geqslant H\left(\frac{\mu}{m}\right) - \left|H'\left(\frac{k\alpha+2\mu}{2m+n\alpha}\right)\right|\frac{\alpha|\mu n - mk|}{m(2m+n\alpha)}$$
$$\geqslant H\left(\frac{\mu}{m}\right) - H'\left(\frac{\alpha}{2m+n\alpha}\right)\frac{\alpha mn}{m(2m+n\alpha)}$$
$$\geqslant H\left(\frac{\mu}{m}\right) + O\big(m^{-1}n\ln m\big).$$

Therefore

$$\binom{n}{k} P(k,\mu) \leqslant \exp\left[-mH\left(\frac{\mu}{m}\right) + \sigma n \ln n\right].$$

It follows then that

$$\binom{n}{k} P(k,\mu) \leqslant \exp(-\sigma n \ln n),$$

if

$$\frac{2\sigma n \ln n}{\ln(m/n) + O(\ln\ln n)} \leqslant \mu \leqslant \frac{m}{2},$$

thus if $\gamma n \leqslant \mu \leqslant m/2$, for some $\gamma > 0$. (Since $m \leqslant m(n)n$, $\ln(m/n) = O(\ln n)$.) So the total contribution of these summands to the bound of $E[C_n]$ is superexponentially small. Consequently q.s. $MG_\alpha(m,n)$ has no component with the number of edges between γn and $m/2$.

Step 2. Consider $\mu \in [k-1, \gamma n]$. The ratio $\mathcal{R}(\mu,k)$ of the $(\mu+1,k)$ summand to the (μ,k) summand in the bound for $E[C_n]$ is given by

$$\mathcal{R}(\mu,k) = \frac{m-\mu}{\mu+1} \cdot \frac{(k\alpha+2\mu)(k\alpha+2\mu+1)}{((n-k)\alpha+2(m-\mu)-2)((n-k)\alpha+2(m-\mu)-1)}.$$

For $\mu \leqslant \gamma n$ we have then

$$\mathcal{R}(\mu,k) \leqslant \big(1+O(n/m)\big)R_0(\mu,k); \qquad \mathcal{R}_0(\mu,k) := \frac{(k\alpha+2(\mu+1))^2}{4m(\mu+1)}.$$

Since $x^{-1}(k\alpha+x)^2$ has a single minimum, and decreases (increases) to the left (to the right) of the point of minimum,

$$\max\big\{\mathcal{R}_0(\mu,k)\colon\ k-1 \leqslant \mu \leqslant \gamma n\big\} \leqslant \max\big\{\mathcal{R}_0(k-1,k), \mathcal{R}_0(\gamma n,k)\big\} \leqslant \sigma' \frac{n}{m},$$
$$\sigma' = 2\max\left\{\frac{(\alpha+2)^2}{4}, \frac{(\alpha+2\gamma)^2}{4\gamma}\right\}.$$

Therefore, uniformly for $k \in [2, n-1]$,

$$\sum_{k-1 \leqslant \mu \leqslant \gamma n} \binom{n}{k} P(k,\mu) \sim \binom{n}{k} P(k,k-1).$$

Furthermore

$$\frac{\binom{n}{k+1}P(k+1,k)}{\binom{n}{k}P(k,k-1)}$$
$$= \frac{(n-k)(m-k+1)}{(k+1)k} \cdot \frac{((k+1)\alpha+2k-2)((k+1)\alpha+2k-1)}{((n-k)\alpha+2(m-k)+2)((n-k)\alpha+2(m-k)+2)}$$
$$\times \prod_{j=0}^{2(k-1)-1} \frac{(k+1)\alpha+j}{k\alpha+j} \cdot \prod_{j=0}^{2(m-k)-1} \frac{(n-k-1)\alpha+j}{(n-k)\alpha+j}.$$

In the top line the first fraction is of order $mn(k+1)^{-2}$ and the second fraction is of order $(k+1)^2 m^{-2}$, so their product is of order n/m. Next

$$\sum_{j=1}^{2k} \frac{\alpha}{k\alpha+j} \leqslant \int_0^{2k} \frac{\alpha}{k\alpha+x}\,dx = \alpha \ln(\alpha+2),$$

hence the first product in the bottom line is $(\alpha+2)^\alpha$ at most, while the second product is 1 at most. Therefore the overall product is of order n/m. We conclude that

$$\sum_{2\leqslant k\leqslant n-1} \binom{n}{k} P(k,k-1) \sim \binom{n}{2} P(2,1) = \binom{n}{2}\binom{m}{1} \frac{(2\alpha)_2((n-1)\alpha)_{2(m-1)}}{(n\alpha)_{2m}}.$$

Here (see the proof of Lemma 4 (4.4))

$$\frac{((n-1)\alpha)_{2(m-1)}}{(n\alpha)_{2m}} \sim \frac{(n\alpha)^\alpha}{(n\alpha+2m)^{2+\alpha}}.$$

Since $\liminf m/n^{1+1/\alpha} > 0$, we see that

$$\sum_{2\leqslant k\leqslant n-1} \binom{n}{k} P(k,k-1) \leqslant_b \frac{n^{\alpha+2}}{m^{1+\alpha}} \leqslant_b n^{-1/\alpha} \to 0.$$

Thus $\mathbf{Pr}(A_n^c) \to 0$, if $m/m(n) \in [a,n]$.

Recalling that $X_n = O_p(1)$ for such m, we obtain that whp $MG_\alpha(n,m)$ consists of a giant component and possibly few isolated vertices. Consequently, under the condition (7.2),

$$\lim_{n,m\to\infty} \mathbf{Pr}\bigl\{MG_\alpha(n,m) \text{ is connected}\bigr\} = \lim_{n,m\to\infty} \mathbf{Pr}\{X_n = 0\} = e^{-x}.$$

Since $e^{-x} \to 1$ as $x \to 0$, we conclude that

$$\lim_{n,m\to\infty} \mathbf{Pr}\bigl\{MG_\alpha(n,m) \text{ is connected}\bigr\} = 1,$$

provided that $\liminf m/m(n) = \infty$. Theorem 2 is proved completely. $\square$

Acknowledgments

I am very much indebted to Laci Lovász for posing an interesting problem. My thanks to Alan Frieze for his interest and kind encouragement. I am grateful to an excellent referee who subjected the original paper and its revision to two rounds of close scrutiny, and sent me several pages of penetrating critical comments and helpful suggestions. Thank you for your time and extraordinary care.

Appendix A. Proof of Lemma 4(c) (4.4)

By (4.1)–(4.2),

$$\frac{E_{k_1,k_2}}{E_{k_1}E_{k_2}} = R(k_1,k_2) := (n\alpha)_{2m}\frac{\langle n\rangle_k\langle m\rangle_{k-2}((n-k)\alpha)_{2(m-k+2)}}{\prod_{i=1}^2\langle n\rangle_{k_i}\langle m\rangle_{k_i-1}((n-k_i)\alpha)_{2(m-k_i+1)}}.$$

In particular,

$$R(1,k_2) = \frac{n-k_2}{n}\frac{(n\alpha)_{2m}}{((n-1)\alpha)_{2m}}\frac{((n-k_2-1)\alpha)_{2(m-k_2+1)}}{((n-k_2)\alpha)_{2(m-k_2+1)}}.$$

Here, by $x\Gamma(x) = \Gamma(x+1)$,

$$\frac{(n\alpha)_{2m}}{((n-1)\alpha)_{2m}} = \frac{\Gamma((n-1)\alpha)}{\Gamma(n\alpha)}\frac{\Gamma(n\alpha+2m)}{\Gamma((n-1)\alpha+2m)},$$
$$\frac{((n-k_2-1)\alpha)_{2(m-k_2+1)}}{((n-k_2)\alpha)_{2(m-k_2+1)}} = \frac{\Gamma((n-k_2)\alpha)}{\Gamma((n-k_2-1)\alpha)}\frac{\Gamma((n-k_2-1)\alpha+2(m-k_2+1))}{\Gamma((n-k_2)\alpha+2(m-k_2-1))}.$$

Applying a simple asymptotic formula (Bateman and Erdélyi [5])

$$\frac{\Gamma(z+x_1)}{\Gamma(z+x_2)} = z^{x_1-x_2}\bigl(1+O\bigl(z^{-1}\bigr)\bigr), \quad z\to\infty, \tag{A.1}$$

to each of the four fractions, we obtain

$$R(1,k_2) = \bigl(1-k_2/n+O(1/n)\bigr)\left(\frac{n\alpha+2m}{n\alpha}\right)^{\alpha}\left(\frac{(n-k_2)\alpha}{(n-k_2)\alpha+2(m-k_2)}\right)^{\alpha}$$
$$= 1+O(k_2/n) = e^{O(k_2/n)}.$$

The relation (4.4) will follow if we show that

$$Q(k_1,k_2) := \frac{R(k_1+1,k_2)}{R(k_1,k_2)} = \exp\left(\frac{2k_2}{n}\bigl(\beta+O(k/n)\bigr)+O(1/n)\right). \tag{A.2}$$

By the definition of $R(k_1,k_2)$,

$$Q(k_1,k_2) = \frac{n-k}{n-k_1}\frac{m-k+2}{m-k_1+1}\frac{((n-k_1)\alpha)_{2(m-k_1+1)}}{((n-k_1-1)\alpha)_{2(m-k_1)}}\frac{((n-k-1)\alpha)_{2(m-k+1)}}{((n-k)\alpha)_{2(m-k+2)}}.$$

Here

$$\frac{n-k}{n-k_1} = \left(1 + \frac{k-k_1}{n-k}\right)^{-1} = \left(1 + \frac{k_2}{n-k}\right)^{-1} = \exp\left(-\frac{k_2}{n} + O\left(k_2 k/n^2\right)\right), \tag{A.3}$$

and likewise

$$\frac{m-k+2}{m-k_1+1} = \exp\left(-\frac{k_2-1}{m} + O\left(k_2 k/n^2\right)\right). \tag{A.4}$$

Further

$$\begin{aligned} &\frac{((n-k_1)\alpha)_{2(m-k_1+1)}}{((n-k_1-1)\alpha)_{2(m-k_1)}} \\ &\quad = \frac{((n-k_1)\alpha)_{2(m-k_1)}}{((n-k_1-1)\alpha)_{2(m-k_1)}} \cdot \left((n-k_1)\alpha + 2(m-k_1)\right)^2 e^{O(1/n)}. \end{aligned} \tag{A.5}$$

Invoking the Gamma function substitution for the fraction on the right and using (A.1) again, we obtain

$$\frac{((n-k_1)\alpha)_{2(m-k_1)}}{((n-k_1-1)\alpha)_{2(m-k_1)}} = \left[\frac{(n-k_1)\alpha + 2(m-k_1)}{(n-k_1)\alpha}\right]^{\alpha} \cdot e^{O(1/n)}. \tag{A.6}$$

Similarly

$$\begin{aligned} &\frac{((n-k-1)\alpha)_{2(m-k+1)}}{((n-k)\alpha)_{2(m-k+2)}} \\ &\quad = \frac{((n-k-1)\alpha)_{2(m-k+1)}}{((n-k)\alpha)_{2(m-k+1)}} \cdot \left((n-k)\alpha + 2(m-k)\right)^{-2} e^{O(1/n)}, \end{aligned} \tag{A.7}$$

where

$$\frac{((n-k-1)\alpha)_{2(m-k+1)}}{((n-k)\alpha)_{2(m-k+1)}} = \left[\frac{(n-k)\alpha}{(n-k)\alpha + 2(m-k)}\right]^{\alpha} \cdot e^{O(1/n)}. \tag{A.8}$$

So, combining (A.3)–(A.8), we have

$$\begin{aligned} Q(k_1, k_2) &= \exp\left(-(\alpha+1)\frac{k_2}{n} - \frac{k_2}{m} + O(k_2 k/n) + O\left(n^{-1}\right)\right) \\ &\quad \times \left[\frac{(n-k_1)\alpha + 2(m-k_1)}{(n-k)\alpha + 2(m-k)}\right]^{\alpha+2}; \end{aligned}$$

here

$$\begin{aligned}\frac{(n-k_1)\alpha+2(m-k_1)}{(n-k)\alpha+2(m-k)} &= 1+\frac{k_2}{n}\frac{\alpha+2}{\alpha+c}+O\big(kk_2/n^2\big)\\ &= \exp\left(\frac{k_2}{n}\frac{\alpha+2}{\alpha+c}+O\big(kk_2/n^2\big)\right).\end{aligned}$$

Therefore

$$Q(k_1,k_2)=\exp\left[\frac{k_2}{n}\left(-(\alpha+1)-\frac{2}{c}+\frac{(\alpha+2)^2}{\alpha+c}\right)+O\big(kk_2/n^2\big)+O\big(n^{-1}\big)\right],$$

which proves (A.2), since the factor of k_2/n is exactly $2\beta(c)$, cf. (4.3).

References

[1] D.J. Aldous, B. Pittel, On a random graph with immigrating vertices: Emergence of the giant component, Random Structures Algorithms 17 (2000) 79–102.
[2] N. Alon, J. Spencer, The Probabilistic Method, John Wiley and Sons, New York, 1992.
[3] A.-L. Barabási, R. Albert, Emergence of scaling in random networks, Science 286 (1999) 509–512.
[4] A.-L. Barabási, R. Albert, H. Jeong, Scale-free characteristics of random networks: The topology of the world-wide web, Physica A 281 (2000) 69–77.
[5] H. Bateman, A. Erdélyi, Higher Transcendental Functions, I, McGraw–Hill, New York, 1953.
[6] E.A. Bender, E.R. Canfield, The asymptotic number of labelled graphs with given degree sequences, J. Combin. Theory Ser. A 24 (1978) 296–307.
[7] F. Bergeron, P. Flajolet, B. Salvy, Varieties of increasing trees, in: J. Raoult (Ed.), Lecture Notes in Comput. Sci., Springer-Verlag, Berlin, 1992.
[8] B. Bollobás, The evolution of random graphs, Trans. Amer. Math. Soc. 286 (1984) 257–274.
[9] B. Bollobás, Random Graphs, 2nd edition, Cambridge Univ. Press, Cambridge, 2001.
[10] B. Bollobás, C. Borgs, J. Chayes, O. Riordan, Directed scale-free graphs, in: Proceedings of the 14-th ACM–SIAM Symposium on Discrete Algorithms (SODA), 2003, pp. 132–139.
[11] B. Bollobás, O. Riordan, J. Spencer, G. Tusnádi, The degree sequence of a scale-free random graph, Random Structures Algorithms 18 (2001) 279–290.
[12] B. Bollobás, O. Riordan, The diameter of a scale-free random graph, Combinatorica 4 (2004) 5–34.
[13] C. Borgs, J.T. Chayes, R. van der Hofstad, G. Slade, J. Spencer, Random subgraphs of finite graphs: I. The scaling window under the triangle condition, Random Structures Algorithms 37 (2005) 137–184.
[14] C. Borgs, J.T. Chayes, R. van der Hofstad, G. Slade, J. Spencer, Random subgraphs of finite graphs: II. The lace expansion and the triangle condition, Ann. Probab. 33 (2005) 1886–1944.
[15] C. Cooper, A.M. Frieze, A general model of undirected web graphs, Random Structures Algorithms 22 (2003) 311–335.
[16] R. Durrett, Random Graph Dynamics, Cambridge Univ. Press, Cambridge, 2006.
[17] P. Erdős, A. Rényi, On the evolution of random graphs, Publ. Math. Hungar. Acad. Sci. 5 (1960) 17–61.
[18] P.J. Flory, Principles of Polymer Chemistry, Cornell Univ. Press, Ithaca, NY, 1953.
[19] S. Janson, T. Łuczak, A. Ruciński, Random Graphs, John Wiley and Sons, New York, 2000.
[20] S. Janson, D.E. Knuth, T. Łuczak, B. Pittel, The birth of the giant component, Random Structures Algorithms 4 (1993) 233–358.
[21] D.E. Knuth, R. Motwani, B. Pittel, Stable husbands, Random Structures Algorithms 1 (1990) 1–14.
[22] T. Łuczak, Component behavior near the critical point of the random graph process, Random Structures Algorithms 1 (1990) 287–310.
[23] T. Łuczak, On the number of sparse connected graphs, Random Structures Algorithms 1 (1990) 171–173.
[24] T. Łuczak, B. Pittel, J.C. Wierman, The structure of random graph near the point of phase transition, Trans. Amer. Math. Soc. 341 (1994) 721–748.
[25] H. Mahmoud, R.T. Smythe, J. Szymański, On the structure of random plane-oriented recursive trees and their branches, Random Structures Algorithms 4 (1993) 151–176.

[26] B.D. McKay, Asymptotics for symmetric 0–1 matrices with prescribed row sums, Ars Combin. 19A (1985) 15–25.
[27] B.D. McKay, N.C. Wormald, Asymptotic enumeration by degree sequence of graphs with degrees $o(n^{1/2})$, Combinatorica 11 (1991) 369–382.
[28] M. Molloy, B. Reed, A critical point for random graphs with a given degree sequence, Random Structures Algorithms 6 (1995) 161–179.
[29] M. Molloy, B. Reed, The size of the giant component of a random graph with a given degree sequence, Combin. Probab. Comput. 7 (1998) 295–305.
[30] J.W. Moon, Counting Labelled Trees, Canadian Mathematics Congress, Montreal, 1970.
[31] A. Nachmias, Y. Peres, Critical percolation on random regular graphs, Random Structures Algorithms, in press.
[32] B. Pittel, Note on the heights of random recursive trees and random m-ary search trees, Random Structures Algorithms 5 (1994) 337–347.
[33] B. Pittel, Edge percolation on a random regular graph of low degree, Ann. Probab. 36 (2008) 1359–1389.
[34] B. Pittel, W. Woyczynski, Random tree-type partitions as a model of acyclic polymerization, II. Holtsmark $3/2$-stable distribution of the supercritical gel, Ann. Probab. 18 (1990) 319–341.
[35] B. Pittel, W. Woyczynski, A graph-valued Markov process as 'rings allowed' polymerization model: Subcritical behavior, SIAM J. Appl. Math. 50 (1990) 1200–1220.
[36] B. Pittel, W. Woyczynski, J. Mann, Random tree-type partitions as a model of a cyclic polymerization, I. Gaussian behavior of the subcritical sol phase, in: J. Jaworski, M. Karoński, A. Ruciński (Eds.), Random Graphs '87, John Wiley and Sons, 1990, pp. 223–273.
[37] J.L. Spouge, Polymers and random graphs: Asymptotic equivalence to branching processes, J. Stat. Phys. 38 (1985) 573–587.
[38] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Univ. Press, Cambridge, 1999.
[39] W.H. Stockmayer, Theory of molecular size distribution and gel formation in branched chain polymers, J. Chem. Phys. 11 (1943) 45–55.
[40] W.H. Stockmayer, Theory of molecular size distribution and gel formation in branched polymers. II. General crosslinking, J. Chem. Phys. 12 (1944) 125–131.
[41] D.J. Watts, Small Worlds, Princeton Univ. Press, Princeton, 1999.
[42] D. Watts, S. Strogatz, Collective dynamics of small-world networks, Nature 393 (1998) 440–442.
[43] P. Whittle, Systems in Stochastic Equilibrium, John Wiley and Sons, New York, 1986.
[44] N.C. Wormald, Differential equations for random processes and random graphs, Ann. Appl. Probab. 5 (1995) 1217–1235.
[45] E.M. Wright, The number of connected sparsely edged graphs. III. Asymptotic results, J. Graph Theory 4 (1980) 393–407.